

Regularity in the Obstacle Problem for Parabolic Non-divergence Operators of Hörmander type

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October 17, 2012

Abstract

In this paper we continue the study initiated in [FGN12] concerning the obstacle problem for a class of parabolic nondivergence operators structured on a set of vector fields $X = \{X_1, \dots, X_q\}$ in \mathbf{R}^n with C^∞ -coefficients satisfying Hörmander's finite rank condition, i.e., the rank of $\text{Lie}[X_1, \dots, X_q]$ equals n at every point in \mathbf{R}^n . In [FGN12] we proved, under appropriate assumptions on the operator and the obstacle, the existence and uniqueness of strong solutions to a general obstacle problem. The main result of this paper is that we establish further regularity, in the interior as well as at the initial state, of strong solutions. Compared to [FGN12] we in this paper assume, in addition, that there exists a homogeneous Lie group $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$ such that X_1, \dots, X_q are left translation invariant on \mathbf{G} and such that X_1, \dots, X_q are δ_λ -homogeneous of degree one.

2000 *Mathematics Subject classification.* 35K70, 35B65, 35B44, 35A09

Keywords and phrases: obstacle problem, parabolic equations, Hörmander condition, hypoelliptic, regularity.

1 Introduction

In this paper we consider the obstacle problem for a class of second order parabolic subelliptic partial differential equations in non-divergence form and modeled on a system of vector fields satisfying the Hörmander's finite rank condition. In particular, we consider operators

$$\mathcal{H} = \sum_{i,j=1}^q a_{ij}(x, t) X_i X_j + \sum_{i=1}^q b_i(x, t) X_i - \partial_t, \quad (1.1)$$

where $(x, t) \in \mathbf{R}^{n+1}$, q is a positive integer and the functions $\{a_{ij}(\cdot, \cdot)\}$ and $\{b_i(\cdot, \cdot)\}$ are bounded and measurable on \mathbf{R}^{n+1} . In the following, and throughout the paper, we by $D \subset \mathbf{R}^{n+1}$ denote a bounded and open cylindrical domain of the form $D = \Omega \times (T_1, T_2]$ where $\Omega \subset \mathbf{R}^n$ is a bounded domain and $-\infty < T_1 < T_2 < \infty$. We let $\partial_p D$ denote the parabolic boundary of D . Let $g, f, \varphi : \bar{D} \rightarrow \mathbf{R}^{n+1}$ be such that $g \geq \varphi$ on \bar{D} and assume that g, f, φ are continuous and bounded on \bar{D} . We consider the following obstacle problem,

$$\begin{cases} \max\{\mathcal{H}u(x, t) - f(x, t), \varphi(x, t) - u(x, t)\} = 0, & \text{in } D, \\ u(x, t) = g(x, t), & \text{on } \partial_p D. \end{cases} \quad (1.2)$$

The purpose of the paper is to advance the mathematical theory for the obstacle problem in (1.2) and in particular to continue the study of the obstacle problem initiated in [FGN12] where a number of important

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steps were taken towards developing a rigorous existence theory for the problem in (1.2). The main result in [FGN12] is the existence and uniqueness of a strong solution to the problem in (1.2) in certain bounded cylindrical domains D . Recall that we say that $u \in \mathcal{S}_{X,\text{loc}}^1(D) \cap C(\overline{D})$ is a strong solution to problem (1.2) if the differential inequality is satisfied a.e. in D and the boundary data is attained at any point of $\partial_p D$. Here $C(\overline{D})$ is the space of functions, continuous on \overline{D} , and the Sobolev-Stein space $\mathcal{S}_{X,\text{loc}}^1(D)$ is defined in Section 2. The subscript X indicates that the space is defined with respect to the system $X = \{X_1, \dots, X_q\}$. Moreover, while the study in [FGN12] focused on existence and uniqueness results, the main purpose of this paper is to establish further regularity for solutions to the problem (1.2). In this paper we will, compared to [FGN12], impose more restrictive assumptions on the system $X = \{X_1, \dots, X_q\}$. To discuss the structural assumptions imposed on the operator \mathcal{H} , as well as the regularity assumptions on a_{ij} , b_i , f , φ and g , we first note that we assume that the system $X = \{X_1, \dots, X_q\}$ is a set of vector fields in \mathbf{R}^n with C^∞ -coefficients, i.e., $X_i = (c_{i1}(x), \dots, c_{in}(x)) \cdot \nabla$, $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$, where $c_{ik} \in C^\infty(\mathbf{R}^n)$ and \cdot denotes scalar product in \mathbf{R}^n . For technical reasons we will assume that c_{ik} are bounded; however, this is not restrictive, since the obstacle problem is stated in a bounded domain. In this paper we impose two essential restrictions on the system $X = \{X_1, \dots, X_q\}$. Firstly, we assume that there exists a homogeneous Lie group $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$, we refer the reader to Section 2 for the definition of \mathbf{G} , such that

$$\begin{aligned} (i) & X_1, \dots, X_q \text{ are left translation invariant on } \mathbf{G}, \\ (ii) & X_1, \dots, X_q \text{ are } \delta_\lambda\text{-homogeneous of degree one.} \end{aligned} \quad (1.3)$$

Secondly, we assume that

$$\text{the vector fields } X = \{X_1, \dots, X_q\} \text{ satisfy Hörmanders finite rank condition.} \quad (1.4)$$

To be more precise, recall that the Lie-bracket between two vector fields X_i and X_j is defined as $[X_i, X_j] = X_i X_j - X_j X_i$ and for an arbitrary multiindex $\theta = (\theta_1, \dots, \theta_d)$, $d \geq 1$, we define

$$X^\theta = [X_{\theta_d}, [X_{\theta_{d-1}}, \dots [X_{\theta_2}, X_{\theta_1}]]].$$

The system $X = \{X_1, \dots, X_q\}$ is said to satisfy a Hörmander finite rank condition of order s if $s < \infty$ is the least integer for which the vector space spanned by $\{X_i^\theta(x) : i = 1, \dots, q, |\theta| \leq s\}$ is \mathbf{R}^n , for all $x \in \mathbf{R}^n$. Note that in [FGN12] we did not assume (1.3). At the end of the paper we discuss to which extent the assumptions in (1.3) are necessary for our main results. Concerning the $q \times q$ matrix-valued function $A = A(x, t) = \{a_{ij}(x, t)\} = \{a_{ij}\}$ we assume that A is real symmetric with bounded and measurable entries and that

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^q a_{ij}(x, t) \xi_i \xi_j \leq \Lambda |\xi|^2 \text{ whenever } (x, t) \in \mathbf{R}^{n+1}, \xi \in \mathbf{R}^q, \quad (1.5)$$

for some Λ , $1 \leq \Lambda < \infty$. Let $d_p((x, t), (y, s)) = (d(x, y)^2 + |t - s|)^{1/2}$ where $d(x, y)$ is the Carnot-Carathéodory distance, between $x, y \in \mathbf{R}^n$, induced by $\{X_1, \dots, X_q\}$, see Section 2.1. Concerning the regularity of a_{ij} and b_i we will assume that a_{ij} and b_i have further regularity beyond being only bounded and measurable. In fact, we assume that

$$a_{ij}, b_i \in \mathcal{C}_X^{0,\alpha}(D) \text{ whenever } i, j \in \{1, \dots, q\}, \quad (1.6)$$

where $\mathcal{C}_X^{0,\alpha}(D)$ is the space of functions which are bounded and Hölder continuous on D and where the Hölder continuity is defined in terms of the (parabolic) distance induced by the vector fields. In particular, we assume that there exists a constant c , $0 < c < \infty$, such that

$$|a_{ij}(x, t) - a_{ij}(y, s)| + |b_i(x, t) - b_i(y, s)| \leq c(d_p((x, t), (y, s)))^\alpha,$$

whenever $(x, t), (y, s) \in D$, $i, j \in \{1, \dots, q\}$.

We are now ready to formulate the main results proved in this paper. We refer to Section 2.2 for the definition of the Hölder spaces $\mathcal{C}_X^{m,\alpha}$, $m \in \{0, 1, 2\}$, $\alpha \in (0, 1)$, and Sobolev-Stein spaces \mathcal{S}_X^p , $1 \leq p \leq \infty$. When we in the following write that a constant c depends on the operator \mathcal{H} , $c = c(\mathcal{H})$, we mean that the constant c only depends on n , q , $X = \{X_1, \dots, X_q\}$, Λ and \mathbf{c}_α where

$$\mathbf{c}_\alpha = \sum_{i,j=1}^q \|a_{ij}\|_{\mathcal{C}_X^{0,\alpha}(D)} + \sum_{i=1}^q \|b_i\|_{\mathcal{C}_X^{0,\alpha}(D)}.$$

Given bounded domains Ω, Ω' in \mathbf{R}^n , and $T > 0$, we let $\Omega_T = \Omega \times (0, T]$, $\Omega'_T = \Omega' \times (0, T]$. We prove the following two theorems.

Theorem 1.1 *Let \mathcal{H} be defined as in (1.1), assume (1.3)-(1.6), let Ω, Ω' be bounded domains in \mathbf{R}^n such that $\Omega' \subset \subset \Omega$ and let $0 < T'' < T' < T$. Let $g, f, \varphi : \bar{\Omega}_T \rightarrow \mathbf{R}^{n+1}$ be such that $g \geq \varphi$ on $\bar{\Omega}_T$ and assume that g, f, φ are continuous and bounded on $\bar{\Omega}_T$. Let $\alpha \in (0, 1)$ and let u be a strong solution to problem (1.2) in Ω_T . Then the following holds:*

i) if $\varphi \in \mathcal{C}_X^{0,\alpha}(\Omega_T)$ then $u \in \mathcal{C}_X^{0,\alpha}(\Omega' \times (T'', T'))$ and

$$\|u\|_{\mathcal{C}_X^{0,\alpha}(\Omega' \times (T'', T'))} \leq c \left(\alpha, \Omega, \Omega', T, T', T'', \mathcal{H}, \|f\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)}, \|g\|_{L^\infty(\Omega_T)}, \|\varphi\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)} \right);$$

ii) if $\varphi \in \mathcal{C}_X^{1,\alpha}(\Omega_T)$ then $u \in \mathcal{C}_X^{1,\alpha}(\Omega' \times (T'', T'))$ and

$$\|u\|_{\mathcal{C}_X^{1,\alpha}(\Omega' \times (T'', T'))} \leq c \left(\alpha, \Omega, \Omega', T, T', T'', \mathcal{H}, \|f\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)}, \|g\|_{L^\infty(\Omega_T)}, \|\varphi\|_{\mathcal{C}_X^{1,\alpha}(\Omega_T)} \right);$$

iii) if $\varphi \in \mathcal{C}_X^{2,\alpha}(\Omega_T)$ then $u \in \mathcal{S}_X^\infty(\Omega' \times (T'', T'))$ and

$$\|u\|_{\mathcal{S}_X^\infty(\Omega' \times (T'', T'))} \leq c \left(\alpha, \Omega, \Omega', T, T', T'', \mathcal{H}, \|f\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)}, \|g\|_{L^\infty(\Omega_T)}, \|\varphi\|_{\mathcal{C}_X^{2,\alpha}(\Omega_T)} \right).$$

Theorem 1.2 *Let \mathcal{H} be defined as in (1.1), assume (1.3)-(1.6), let Ω, Ω' be bounded domains in \mathbf{R}^n such that $\Omega' \subset \subset \Omega$ and let $0 < T' < T$. Let $g, f, \varphi : \bar{\Omega}_T \rightarrow \mathbf{R}^{n+1}$ be such that $g \geq \varphi$ on $\bar{\Omega}_T$ and assume that g, f, φ are continuous and bounded on $\bar{\Omega}_T$. Let $\alpha \in (0, 1)$ and let u be a strong solution to problem (1.2) in Ω_T . Then the following holds:*

i) if $g, \varphi \in \mathcal{C}_X^{0,\alpha}(\Omega_T)$ then $u \in \mathcal{C}_X^{0,\alpha}(\Omega'_T)$ and

$$\|u\|_{\mathcal{C}_X^{0,\alpha}(\Omega'_T)} \leq c \left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)}, \|g\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)}, \|\varphi\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)} \right);$$

ii) if $g, \varphi \in \mathcal{C}_X^{1,\alpha}(\Omega_T)$ then $u \in \mathcal{C}_X^{1,\alpha}(\Omega'_T)$ and

$$\|u\|_{\mathcal{C}_X^{1,\alpha}(\Omega'_T)} \leq c \left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)}, \|g\|_{\mathcal{C}_X^{1,\alpha}(\Omega_T)}, \|\varphi\|_{\mathcal{C}_X^{1,\alpha}(\Omega_T)} \right);$$

iii) if $g, \varphi \in \mathcal{C}_X^{2,\alpha}(\Omega_T)$ then $u \in \mathcal{S}_X^\infty(\Omega'_T)$ and

$$\|u\|_{\mathcal{S}_X^\infty(\Omega'_T)} \leq c \left(\alpha, \Omega_T, \Omega'_T, \mathcal{H}, \|f\|_{\mathcal{C}_X^{0,\alpha}(\Omega_T)}, \|g\|_{\mathcal{C}_X^{2,\alpha}(\Omega_T)}, \|\varphi\|_{\mathcal{C}_X^{2,\alpha}(\Omega_T)} \right).$$

Theorem 1.1 concerns the optimal interior regularity for the solution u to the obstacle problem under different assumptions on the regularity of the obstacle φ . In particular, Theorem 1.1 treats the case of non-smooth obstacles as well as the case of smooth obstacles. The results stated in the theorems are similar: the solution is, up to \mathcal{S}_X^∞ -smoothness, as smooth as the obstacle. Note that Theorem 1.2 gives similar results but in this case Ω'_T is not a compact subset of Ω_T . As such Theorem 1.2 concerns the optimal regularity up to the initial state for the solution u to the obstacle problem under the stated assumptions on the regularity of the obstacle φ . Based on the discussion below we claim that Theorem 1.1 and Theorem 1.2 represent two new contributions to the literature on optimal regularity for obstacle problems. In fact, we are not aware of any previous works, in the parabolic and genuinely subelliptic case, i.e., when $X = \{X_1, \dots, X_q\}$ is not identical to $\{\partial_{x_1}, \dots, \partial_{x_n}\}$, devoted to the obstacle problem in (1.2).

To outline previous work on optimal interior regularity in obstacle problems we first discuss the more classical case when $X = \{X_1, \dots, X_q\} \equiv \{\partial_{x_1}, \dots, \partial_{x_n}\}$ and we note that in this case there is an extensive literature on the existence of generalized solutions to the obstacle problem in Sobolev spaces starting with the pioneering papers [McK65], [vM74a], [vM74b] and [Fri75]. Furthermore, optimal regularity of the solution to the obstacle problem for the Laplace equation was first proved by Caffarelli and Kinderlehrer [CK80] and we note that the techniques used in [CK80] are based on the Harnack inequality for harmonic functions and the control of a

harmonic function by its Taylor expansion. The most extensive and complete treatment of the obstacle problem for the heat equation can be found in Caffarelli, Petrosyan and Shahgholian [CPS04] and it is interesting to note that most of the arguments in [CPS04] make use of a blow-up technique previously also used by Caffarelli, Karp and Shahgholian in [CKS00] in the stationary case. We here also mention the paper [BDM06] where the optimal regularity of the obstacle problem for second order uniformly elliptic parabolic equations has been proved by a method inspired by the original one in [CK80] based on the Harnack inequality. On the other hand the blow-up method has been employed in more general settings in [PS07], [Sha08]. Concerning previous work on the optimal interior regularity in obstacle problems in a subelliptic setting we note that in [DGS03] the obstacle problem is considered for the strongly degenerate case of sublaplacians on Carnot groups. The paper [DGP07] addresses, in the same framework, the study of the regularity of the free boundary. In particular, the sublaplacian considered in [DGS03] can be considered as a special case of the stationary versions of the more general operators studied in this paper. Finally, we note that in [FNPP10] the author, together with Nyström, Pascucci and Polidoro, recently have established an appropriate version of Theorem 1.1 for a class of second order differential operators of Kolmogorov type of the form

$$H = \sum_{i,j=1}^m a_{ij}(x,t) \partial_{x_i x_j} + \sum_{i=1}^m b_i(x,t) \partial_{x_i} + \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} - \partial_t \quad (1.7)$$

where $(x,t) \in \mathbf{R}^{n+1}$, m is a positive integer satisfying $m \leq n$, the functions $\{a_{ij}(\cdot, \cdot)\}$ and $\{b_i(\cdot, \cdot)\}$ are continuous and bounded and the matrix $B = \{b_{ij}\}$ is a matrix of constant real numbers. The structural assumptions imposed in [FNPP10] on the operator H implies that H is a hypoelliptic ultraparabolic operator of Kolmogorov type. Note however, that the operator in (1.7) is different from the class of operators considered in this paper due to the fact that space and time couple through the lower order term $Y = \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} - \partial_t$ and since $\{X_1, \dots, X_m\} = \{\partial_{x_1}, \dots, \partial_{x_m}\}$.

To outline previous work on the optimal regularity up to the initial state in obstacle problems we note that there is, already in the case when $X = \{X_1, \dots, X_q\} \equiv \{\partial_{x_1}, \dots, \partial_{x_n}\}$, a very limited literature on this topic. In fact, in this case we are only aware of the results by Nyström [Nys08], Shahgholian [Sha08] (see also Petrosyan and Shahgholian [PS07]). While the arguments in [Sha08] allow for certain fully non-linear parabolic equations, in [Nys08] the techniques were conveyed in the context of pricing of multi-dimensional American options in a financial market driven by a general multi-dimensional Itô diffusion. In [Nys08] the machinery and techniques were developed and described assuming more regularity on the operator and the obstacle than needed and in the standard context of American options. However, the results in [Nys08] and [Sha08] do not apply, for example and considering financial applications, in the setting of Asian options or the Hobson-Rogers model for stochastic volatility [HR98]. Therefore Nyström, Pascucci and Polidoro, in [NPP10], recently established an appropriate version of Theorem 1.2 for the class of second order differential operators of Kolmogorov type briefly defined in (1.7), and we note that [NPP10] is a continuation of the study initialized in [FNPP10]. We also note that the results in [NPP10] also apply to uniformly parabolic equations, i.e., the case when $X = \{X_1, \dots, X_q\} \equiv \{\partial_{x_1}, \dots, \partial_{x_n}\}$. In this case the results in [NPP10] slightly improve upon Theorem 4.3 in [PS07] (see also Theorems 1.2 and 1.3 in [Sha08]) since in [NPP10] the authors get the Hölder regularity of the solution with the optimal exponent.

Concerning our proofs of Theorem 1.1 and Theorem 1.2 we will proceed in a fashion structurally similar to the corresponding proofs in [FNPP10] and [NPP10]. In particular, we will use the type of blow-up technique introduced by Caffarelli, Karp and Shahgholian in [CKS00] in the stationary case and by Caffarelli, Petrosyan and Shahgholian [CPS04] in the study of the heat equation. This method is flexible enough to give stream-lined proofs of the statements in Theorem 1.1 and Theorem 1.2 for non-smooth as well as smooth obstacles. To briefly outline the method we let $d(x,y)$ be the Carnot-Carathéodory distance between $x, y \in \mathbf{R}^n$, induced by $\{X_1, \dots, X_q\}$ and we let $B_d(x,r) = \{y \in \mathbf{R}^n : d(x,y) < r\}$, whenever $x \in \mathbf{R}^n$ and $r > 0$, denote the balls associated to d . We let

$$\begin{aligned} C_r(x,t) &= B_d(x,r) \times (t-r^2, t+r^2), \\ C_r^+(x,t) &= B_d(x,r) \times (t, t+r^2), \\ C_r^-(x,t) &= B_d(x,r) \times (t-r^2, t), \end{aligned} \quad (1.8)$$

whenever $(x,t) \in \mathbf{R}^{n+1}$ and $r > 0$. In fact, we will use a modification of these cylinders, see the discussion below Lemma 3.4. Using the notation in (1.8) we in this paper build the core part of the argument, in the case

of Theorem 1.1, on the function

$$S_k^-(u) = \sup_{C_{2^{-k}}^-(0,0)} |u| \quad (1.9)$$

and, in the case of Theorem 1.2, on the function

$$S_k^+(u) = \sup_{C_{2^{-k}}^+(0,0)} |u|. \quad (1.10)$$

In (1.9) and (1.10) u is a solution to the obstacle problem in $C_1^-(0,0)$ and $C_1^+(0,0)$, respectively. We will assume that the quadruple (u, f, g, φ) , which specifies the degrees of freedom in the obstacle problem, belongs to certain function classes defined in Subsection 2.3. Moreover given φ , in this construction we let, in the proof of Theorem 1.1, F and γ be determined as follows:

$$\begin{aligned} (i) F &= P_0^{(0,0)} \varphi, \quad \gamma = \alpha, \\ (ii) F &= P_1^{(0,0)} \varphi, \quad \gamma = 1 + \alpha, \\ (iii) F &= \varphi, \quad \gamma = 2, \end{aligned} \quad (1.11)$$

where $\alpha \in (0, 1)$ and $P_m^{(0,0)} \varphi$ is a certain intrinsic Taylor expansion associated to φ and as outlined in Subsection 2.2. In particular, as an important step in the proof of Theorem 1.1, we prove that there exists a positive constant c such that, for all $k \in \mathbf{N}$,

$$S_{k+1}^-(u - F) \leq \max \left(c 2^{-(k+1)\gamma}, \frac{S_k^-(u - F)}{2^\gamma}, \frac{S_{k-1}^-(u - F)}{2^{2\gamma}}, \dots, \frac{S_0^-(u - F)}{2^{(k+1)\gamma}} \right). \quad (1.12)$$

In the case of Theorem 1.2, we again assume that (u, f, g, φ) belongs to certain function classes and given g , in this construction, we let F and γ be determined as follows:

$$\begin{aligned} (i) F &= P_0^{(0,0)} g, \quad \gamma = \alpha, \\ (ii) F &= P_1^{(0,0)} g, \quad \gamma = 1 + \alpha, \\ (iii) F &= g, \quad \gamma = 2. \end{aligned} \quad (1.13)$$

Similarly, to prove Theorem 1.2, we prove that there exists a positive constant c such that, for all $k \in \mathbf{N}$,

$$S_{k+1}^+(u - F) \leq \max \left(c 2^{-(k+1)\gamma}, \frac{S_k^+(u - F)}{2^\gamma}, \frac{S_{k-1}^+(u - F)}{2^{2\gamma}}, \dots, \frac{S_0^+(u - F)}{2^{(k+1)\gamma}} \right). \quad (1.14)$$

Note that we are able to carry the argument through using only g , instead of φ , in (1.13) and this is one key aspect of the argument. In either case the proofs of (1.12) and (1.14) are based on arguments by contradiction and the argument in the case of (1.14) differs at key points compared to the corresponding proof in the case (1.12) due to the presence, in the case (1.14), of the boundary at $t = 0$.

The rest of the paper is organized as follows. Section 2 is of preliminary nature and we here introduce relevant notations and recall some basic facts concerning homogeneous Lie groups. We also introduce function spaces and consider their characterizations based on polynomials. In Section 3 we state some results concerning operators \mathcal{H} defined as in (1.1) assuming (1.3)-(1.6). These results concern existence and estimates for fundamental solutions, the Harnack inequality and Schauder estimates. In this section we also use these results to prove a few auxiliary technical estimates to be used in the proofs of Theorem 1.1 and Theorem 1.2. In Section 4 we derive certain estimates for the obstacle problem to be used in the proofs of Theorem 1.1 and Theorem 1.2. Section 5 is devoted to the proof of Theorem 1.1 while Section 6 is devoted to the proof of Theorem 1.2. Finally, in Section 7 we complete this paper with some concluding remarks.

2 Preliminaries

In this section we firstly introduce some notations and recall some basic notions concerning homogeneous Lie groups. We refer to the monograph [BLU07] for a detailed treatment of the subject. Secondly we introduce relevant function spaces and consider their Taylor approximation. Thirdly, and finally, we introduce certain function classes related to the Cauchy-Dirichlet problem and to the obstacle problem. Based on these function classes we are able to give streamlined proofs of important lemmas and to prove Theorem 1.1 and Theorem 1.2 in a unified way.

2.1 Homogeneous Lie groups

Let \circ be a given group law on \mathbf{R}^n and suppose that the map $(x, y) \mapsto y^{-1} \circ x$ is smooth. Then $\mathbf{G} = (\mathbf{R}^n, \circ)$ is called a *Lie group*. On \mathbf{G} we define the left translation operator $\tau_\alpha(x) = \alpha \circ x$ and we say that a vector field X on \mathbf{R}^n is *left-invariant* if $X(\tau_\alpha(x)) = \tau_\alpha(X(x))$. We let \mathfrak{g} denote the Lie algebra of \mathbf{G} , i.e., the Lie-algebra of left-invariant vector fields on \mathbf{R}^n . Moreover, \mathbf{G} is said to be *homogeneous* if there exists a family of *dilations* $(\delta_\lambda)_{\lambda>0}$ on \mathbf{G} of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \dots, x^{(l)}) = (\lambda x^{(1)}, \dots, \lambda^l x^{(l)}) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n), \quad (2.1)$$

where $1 \leq \sigma_1 \leq \dots \leq \sigma_n$, and if $(\delta_\lambda)_{\lambda>0}$ defines an automorphism of the group, i.e.,

$$\delta_\lambda(x \circ y) = (\delta_\lambda(x)) \circ (\delta_\lambda(y)), \quad \text{for all } x, y \in \mathbf{R}^n \text{ and } \lambda > 0.$$

Note that in (2.1) we have that $x^{(i)} \in \mathbf{R}^{n_i}$ for $i \in \{1, \dots, l\}$ and $n_1 + \dots + n_l = n$. In particular, the dilation δ_λ induces a direct sum decomposition on \mathbf{R}^n

$$\mathbf{R}^n = V_1 \oplus \dots \oplus V_l.$$

The natural number

$$Q := \dim V_1 + 2 \dim V_2 + \dots + l \dim V_l = n_1 + 2n_2 + \dots + ln_l, \quad (2.2)$$

is called the *homogeneous dimension* of \mathbf{G} with respect to δ_λ . For short we write $(\mathbf{R}^n, \circ, \delta_\lambda)$ for the Lie group \mathbf{G} equipped with the family of dilations $(\delta_\lambda)_{\lambda>0}$. We next recall some useful facts about homogeneous Lie groups and we first note that (1.3) and (1.4) imply that $n_1 = q$ and that $\text{span}\{X_1(0), \dots, X_q(0)\} = V_1$. Hence it is not restrictive to assume $q = \dim V_1$ and $X_i(0) = \mathbf{e}_i$ for $i = 1, \dots, q$, where $\{\mathbf{e}_i\}_{1 \leq i \leq n}$ denotes the canonical basis of \mathbf{R}^n . In particular, X_i is, for $i = 1, \dots, q$, the unique vector field in \mathfrak{g} that agrees with $\partial/\partial x_i$ at the origin. With these hypotheses $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$ can be referred to as a *homogeneous Carnot group*. We say that \mathbf{G} is of step l and that it has $q = n_1$ generators. In the literature, see, e.g., [Fol75], [HK00], [RS76], [VSC92], a Carnot group (or a stratified group) \mathbf{G} is defined as a connected and simply connected Lie group whose Lie algebra $\tilde{\mathfrak{g}}$ admits a stratification

$$\tilde{\mathfrak{g}} = W_1 \oplus \dots \oplus W_l \text{ with } [W_1, W_i] = W_{i+1} \text{ and } [W_1, W_l] = \{0\}.$$

In fact, any homogeneous Carnot group is a Carnot group according to the classical definition. On the other hand, up to isomorphism, the opposite implication is also true, see, e.g., [BU04]. To continue we set

$$\|x\|_{\mathbf{G}} = \left(\sum_{j=1}^l \sum_{i=1}^{m_j} \left(x_i^{(j)} \right)^{\frac{2m_j}{j}} \right)^{\frac{1}{2l}},$$

and we observe that the above function is homogeneous of degree 1 on \mathbf{R}^n in the sense that

$$\left\| (\lambda x^{(1)}, \dots, \lambda^l x^{(l)}) \right\|_{\mathbf{G}} = \lambda \|x\|_{\mathbf{G}},$$

for every $x \in \mathbf{R}^n$ and for any $\lambda > 0$. In particular, $\|\cdot\|_{\mathbf{G}}$ is a homogeneous norm on \mathbf{G} , smooth away from the origin. Using this homogeneous norm we define a quasi-distance $\hat{d} : \mathbf{R}^n \times \mathbf{R}^n \rightarrow [0, \infty)$ by setting $\hat{d}(x, y) := \|y^{-1} \circ x\|_{\mathbf{G}}$. The term quasi-distance is used to indicate that \hat{d} has the same properties as a distance, except that \hat{d} satisfy the quasi-triangle inequality and not the triangle inequality. We also recall that for every compact set $K \subset \mathbf{R}^n$ there exist two positive constants c_K^- and c_K^+ , such that

$$c_K^- |x - y| \leq \hat{d}(x, y) \leq c_K^+ |x - y|^{\frac{1}{s}}, \quad \text{for all } x, y \in K. \quad (2.3)$$

Here $|\cdot|$ denotes the Euclidean distance and s is the order in Hörmander's finite rank condition for $X = \{X_1, \dots, X_q\}$. For a proof of (2.3) we refer to [NSW85]. Finally, we recall the notion of *Carnot-Carathéodory distance*. In particular, let $X = \{X_1, \dots, X_q\}$ be a system satisfying (1.4) and let $T > 0$. We say that an

absolutely continuous curve $\gamma : [0, T] \rightarrow \mathbf{R}^n$ is a sub-unit curve with respect to the system $X = \{X_1, \dots, X_q\}$, γ is X -subunit for short, if there exist measurable functions $h = (h_1, \dots, h_q)$ such that

$$\gamma'(t) = \sum_{j=1}^q h_j(t) X_j(\gamma(t)) \quad \text{for a.e. } t \in [0, T] \quad \text{with} \quad \sum_{j=1}^q h_j(t)^2 \leq 1 \quad \text{a.e.}$$

In the following we let $l(\gamma)$ denote this T and for $x, y \in \mathbf{R}^n$ we define

$$d(x, y) = \inf \{l(\gamma) \mid \gamma : [0, l(\gamma)] \rightarrow \mathbf{R}^n, \gamma \text{ is } X\text{-subunit}, \gamma(0) = x \text{ and } \gamma(l(\gamma)) = y\}.$$

It is well known by Chow-Rashevsky's connectivity theorem, see [Cho39] and [Ras38], that if $X = \{X_1, \dots, X_q\}$ is a system satisfying (1.4), then the above set is nonempty and hence $d(x, y)$ is finite for every pair of points. In particular, d is referred to as the Carnot-Carathéodory distance, CC -distance for short, between the points x and y . Moreover, d is a metric on \mathbf{R} and $\|x\| := d(0, x)$ is a symmetric, homogeneous norm on \mathbf{G} . Comparing \hat{d} and d one can prove that there exist positive constants c_1, c_2 such that

$$c_1 \hat{d}(x, y) \leq d(x, y) \leq c_2 \hat{d}(x, y) \quad \text{for all } x, y \in \mathbf{R}^n. \quad (2.4)$$

In fact, both \hat{d} and d are homogeneous of degree 1 with respect to δ_λ and on a homogeneous group all homogeneous norms are equivalent, see for instance Chapter 5 in [BLU07]. We next develop the necessary parabolic notation. In particular, given $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$ as above we extend this Lie group to a Lie group on \mathbf{R}^{n+1} by defining

$$(x, t)^{-1} \circ (y, s) = (x^{-1} \circ y, s - t)$$

whenever $(x, t), (y, s) \in \mathbf{R}^{n+1}$. Furthermore, we extend δ_λ to \mathbf{R}^{n+1} by defining $\delta_\lambda(x, t) = (\delta_\lambda(x), \lambda^2 t)$ and we write \mathbf{L} for the constructed homogeneous Lie group $(\mathbf{R}^{n+1}, \circ, \delta_\lambda)$. Notationally we do not differentiate between dilations on \mathbf{G} and \mathbf{L} respectively, however, it will be clear from the context which type of dilations that is used. Moreover, we set

$$\|(x, t)\|_p = d_p((0, 0), (x, t)), \quad \|(x, t)\|_{\mathbf{L}} = (\|x\|_{\mathbf{G}}^{2/l} + |t|^{1/l})^{\frac{1}{2/l}},$$

and we observe that these functions are homogeneous of degree 1 on \mathbf{R}^{n+1} , in the sense that

$$\|\delta_\lambda(x, t)\|_p = \lambda \|(x, t)\|_p, \quad \|\delta_\lambda(x, t)\|_{\mathbf{L}} = \lambda \|(x, t)\|_{\mathbf{L}},$$

for every $(x, t) \in \mathbf{R}^{n+1}$ and for any $\lambda > 0$. Finally, we extend \hat{d} and d to $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ by defining

$$d_p((x, t), (y, s)) = \sqrt{d(x, y)^2 + |t - s|}, \quad \hat{d}_p((x, t), (y, s)) = \sqrt{\hat{d}(x, y)^2 + |t - s|} \quad (2.5)$$

for $(x, t), (y, s) \in \mathbf{R}^{n+1}$. We continue and introduce the appropriate balls. For d we define $B_d(x, r) := \{y \in \mathbf{R}^n : d(x, y) < r\}$, whenever $x \in \mathbf{R}^n$ and $r > 0$ and similarly, for d_p , we define $B_{d_p}((x, t), r) := \{(y, s) \in \mathbf{R}^{n+1} : d_p((x, t), (y, s)) < r\}$. Based on $B_d(x, r)$ we introduce the bounded cylindrical domains $C_r(x, t)$, $C_r^+(x, t)$, $C_r^-(x, t)$ as in (1.8). Then, using the homogeneity of the group \mathbf{G} we see that there exists a constant c such that

$$c^{-1} r^Q \leq |B_d(x, r)| \leq c r^Q \quad \text{and} \quad c^{-1} r^{Q+2} \leq |C_r(x, t)| \leq c r^{Q+2} \quad (2.6)$$

whenever $(x, t) \in \mathbf{R}^{n+1}$ and $r > 0$. Here $|B_d(x, r)|$ and $|C_r(x, t)|$ denote the Euclidean volume of $B_d(x, r)$ and $C_r(x, t)$ respectively and Q is the homogeneous dimension of \mathbf{G} as introduced in (2.2). Throughout the paper we will often write C , C^+ and C^- for the cylinders $C_1(0, 0)$, $C_1^+(0, 0)$, $C_1^-(0, 0)$ introduced in (1.8). Similarly, we will often write C_r , C_r^+ and C_r^- for the cylinders $C_r(0, 0)$, $C_r^+(0, 0)$, $C_r^-(0, 0)$.

Finally we define the exponential map $Exp : \mathbf{g} \rightarrow \mathbf{G}$. For smooth vector fields X , there exists a unique solution to

$$\begin{cases} \gamma'(t) = XI(\gamma(t)) \\ \gamma(0) = x \end{cases}$$

where I is the identity map. We denote this solution $\gamma_X(t)$ and we define $\exp(t \cdot X)(x) := \gamma_X(t)$. Then \exp is well defined for all $X \in \mathbf{g}$, $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. The exponential map $Exp : \mathbf{g} \rightarrow \mathbf{G}$ is defined through

$$Exp(X) := \exp(1 \cdot X)(0).$$

2.2 Function spaces and polynomial approximation

Let $D \subset \mathbf{R}^{n+1}$ be an open domain. We say that D is a bounded cylindrical domain if

$$D = \Omega \times (T_1, T_2] \text{ for some } \Omega \subset \mathbf{R}^n, \text{ open and bounded, and some } -\infty < T_1 < T_2 < \infty. \quad (2.7)$$

Let $\alpha \in (0, 1)$ and let $D \subset \mathbf{R}^{n+1}$ be a bounded cylindrical domain in the sense of (2.7). Let $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$ and $X = \{X_1, \dots, X_q\}$ be as in (1.3) and (1.4). Let \mathbf{L} be the parabolic extension of \mathbf{G} as defined in the previous subsection and let d_p be the parabolic CC-distance introduced in (2.5). In the following we will often denote points in \mathbf{R}^{n+1} by z and ζ , i.e., $z = (x, t)$, $\zeta = (y, s)$. Using this notation we in the following by $\mathcal{C}_X^{0,\alpha}(D)$, $\mathcal{C}_X^{1,\alpha}(D)$ and $\mathcal{C}_X^{2,\alpha}(D)$ denote Hölder spaces defined by the following norms:

$$\begin{aligned} \|u\|_{\mathcal{C}_X^{0,\alpha}(D)} &= \sup_D |u| + \sup_{\substack{z, \zeta \in D \\ z \neq \zeta}} \frac{|u(z) - u(\zeta)|}{d_p(z, \zeta)^\alpha}, \\ \|u\|_{\mathcal{C}_X^{1,\alpha}(D)} &= \|u\|_{\mathcal{C}_X^{0,\alpha}(D)} + \sum_{i=1}^q \|X_i u\|_{\mathcal{C}_X^{0,\alpha}(D)} + \sup_{\substack{z, \zeta \in D \\ z \neq \zeta}} \frac{|u(z) - u(\zeta) - \sum_{j=1}^q (z_j - \zeta_j) X_j u(\zeta)|}{d_p(z, \zeta)^{1+\alpha}}, \\ \|u\|_{\mathcal{C}_X^{2,\alpha}(D)} &= \|u\|_{\mathcal{C}_X^{0,\alpha}(D)} + \sum_{i=1}^q \|X_i u\|_{\mathcal{C}_X^{0,\alpha}(D)} + \sum_{i,j=1}^q \|X_i X_j u\|_{\mathcal{C}_X^{0,\alpha}(D)} + \|\partial_t u\|_{\mathcal{C}_X^{0,\alpha}(D)}. \end{aligned}$$

In particular, note that we could just as well define Hölder spaces in terms of \hat{d}_p . Moreover, we let $\mathcal{C}^0(D)$ denote the set of functions which are continuous on D . Note that if $u \in \mathcal{C}_X^{0,\alpha}(D)$, then u is Hölder continuous in the usual sense as we see using (2.3). Let $m \in \{0, 1, 2\}$, $\alpha \in (0, 1)$. If $u \in \mathcal{C}_X^{k,\alpha}(D')$ for every compact subset D' of D , then we write $u \in \mathcal{C}_{X,\text{loc}}^{k,\alpha}(D)$. Furthermore, for $p \in [1, \infty]$ we define the Sobolev-Stein spaces

$$\mathcal{S}_X^p(D) = \{u \in L^p(D) : X_i u, X_i X_j u, \partial_t u \in L^p(D), i, j = 1, \dots, q\}$$

and we let

$$\|u\|_{\mathcal{S}_X^p(D)} = \|u\|_{L^p(D)} + \sum_{i=1}^q \|X_i u\|_{L^p(D)} + \sum_{i,j=1}^q \|X_i X_j u\|_{L^p(D)} + \|\partial_t u\|_{L^p(D)}.$$

Here $L^p(D)$ is the L^p -norm in $\mathbf{G} \equiv \mathbf{R}^{n+1}$ with respect to the Lebesgue measure. If $u \in \mathcal{S}^p(D')$ for every compact subset D' of D , then we write $u \in \mathcal{S}_{X,\text{loc}}^p(D)$.

We next consider the spaces $\mathcal{C}_X^{m,\alpha}(D)$, for $m \in \{0, 1, 2\}$, and we want to understand to which extent these spaces can be characterized using polynomials. The following outline is based on the results in [GL09] and [Bon09]. Recall that δ_λ satisfies (2.1). In the following we rewrite (2.1) as

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n) \quad (2.8)$$

where $1 \leq \sigma_1 \leq \dots \leq \sigma_n < \infty$. Using this notation, yet another example of a δ_λ -homogeneous norm is given by $\|x\|_{\mathbf{G}'} = \sum_{j=1}^n |x_j|^{1/\sigma_j}$ whenever $x \in \mathbf{R}^n$. Furthermore, if $I = (i_1, \dots, i_n)$ is a multi-index, $i_j \in \mathbf{N} \cup \{0\}$, we let

$$|I|_{\delta_\lambda} = \sum_{j=1}^n \sigma_j i_j$$

denote the δ_λ -length of I and we denote, as usual, by $|I| = \sum_{j=1}^n i_j$ the standard length of the multi-index I . A polynomial $p(x)$ in the variable x_1, \dots, x_n is said to have δ_λ -degree m if $p(x) = \sum_{|I|_{\delta_\lambda} \leq m} a_I x^I$. We denote by $\mathbb{P}_{m,\delta_\lambda}$ the class of all polynomials with δ_λ -degree less than or equal to m . Let Ω be any fixed bounded and open neighborhood of the origin and define $\Omega_\mu = \delta_\mu \Omega$ for $\mu > 0$. Given $1 \leq \sigma_1 \leq \dots \leq \sigma_n$ as in (2.8) we define $\Sigma = \{|I|_{\delta_\lambda} : I = (i_1, \dots, i_n) \in (\mathbf{N} \cup \{0\})^n, |I|_{\delta_\lambda} > 0\}$. Below $C^m(\mathbf{G}, \mathbf{R})$ will denote the set of functions $f : \mathbf{G} \rightarrow \mathbf{R}$ for which differentials along vector fields in \mathbf{g} of δ_λ -length $\leq m$ exists. We are now ready to account for how to approximate functions using polynomials. Below, Z_1, \dots, Z_n denotes the Jacobian basis of \mathbf{g} , the Lie algebra of \mathbf{G} and for $I = (i_1, \dots, i_n)$, Z^I denotes the differential operator $Z^I = Z_1^{i_1} \cdots Z_n^{i_n}$. Similarly, for $I = (i_1, \dots, i_k)$, where $i_j \in \{1, \dots, n\}$ for $j = 1, \dots, k$, we define the differential operator $Z_I = Z_{i_1} \cdots Z_{i_k}$. This time we let $|I|_{\delta_\lambda} = \sum_{j=1}^k \sigma_{i_j}$. Furthermore, $B_{\mathbf{G}'}(y, r)$ denotes $\{x \in \mathbf{R}^n : \|y^{-1} \circ x\|_{\mathbf{G}'} < r\}$. For the proof of the following lemma we refer to Theorem 2.8 in [GL09].

Lemma 2.1 *Let $\Omega \subset \mathbf{R}^n$ be open, let $u : \Omega \rightarrow \mathbf{R}$ be a smooth function, $\alpha \in (0, 1)$ and $m \in \Sigma$. Suppose there exists positive constants A and $\rho > 0$ such that for each $B_{\mathbf{G}'}(z, R) \subset \Omega$ and for each $y \in B_{\mathbf{G}'}(z, \rho R)$ there exists a polynomial $P_m^y \in \mathbb{P}_{m, \delta_\lambda}$ such that*

$$|u(x) - P_m^y(x)| \leq A \|y^{-1} \circ x\|_{\mathbf{G}'}^{m+\alpha} \text{ for all } x \in B_{\mathbf{G}'}(y, \rho R). \quad (2.9)$$

Then there exist positive constants C and $\eta > 0$ depending only on A, α, m, ρ and the structure such that

$$|Z^I u(x_1) - Z^I u(x_2)| \leq C \|x_1^{-1} \circ x_2\|_{\mathbf{G}'}^\alpha,$$

for all $|I|_{\delta_\lambda} = m$, $I = (i_1, \dots, i_n)$, and all $x_1, x_2 \in B_{\mathbf{G}'}(y, \eta R)$.

To find Taylor polynomials we use the following theorem, see Theorem 2 in [Bon09].

Theorem 2.2 *Let $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$ be a homogeneous Lie group. Let $y \in \mathbf{G}$ be fixed and assume that $m \in \mathbf{N} \cup \{0\}$ and that $u \in C^{m+1}(\mathbf{G}, \mathbf{R})$. Let $\{Z_1, \dots, Z_n\}$ be the Jacobian basis for \mathfrak{g} , the Lie algebra of \mathbf{G} . Then, for every $x \in \mathbf{G}$ the Taylor polynomial of δ_λ -degree m of u at y is given by*

$$u(x) = u(y) + \sum_{k=1}^m \sum_{\substack{I=(i_1, \dots, i_k) \\ |I|_{\delta_\lambda} \leq m}} \frac{Z_I u(y)}{k!} \zeta_{i_1}(y^{-1} \circ x) \cdots \zeta_{i_k}(y^{-1} \circ x, t) + R_m(x, y), \quad (2.10)$$

where the function $\zeta(h)$ is defined by

$$h = \text{Exp}(\zeta_1(h)Z_1 + \dots + \zeta_n(h)Z_n).$$

Moreover, there exists a constant c depending only on \mathbf{G} and d_p such that

$$|R_m(x)| \leq \sum_{k=1}^{m+1} \frac{c^k}{k!} \sum_{\substack{I=(i_1, \dots, i_k) \\ |I|_{\delta_\lambda} > m}} d_p(x, y)^{|I|_{\delta_\lambda}} \sup_{\|\xi\|_{\mathbf{L}} \leq c \|y^{-1} \circ x\|_{\mathbf{L}}} |Z_I u(y \circ \xi)|. \quad (2.11)$$

Note that, in [Bon09], Bonfiglioli uses the distance \hat{d}_p in (2.11). This theorem naturally extend to \mathbf{L} , and we remark that $\zeta_i(h) = t$ while $\zeta_i(h)$ is independent of t for $i \in \{1, \dots, n\}$. We also mention that although Lemma 2.1 and Theorem 2.2 are stated using Z^I and Z_I respectively Proposition 20.1.4 in [BLU07] shows, with slight abuse of notation, that $Z_I \in \text{span}\{Z^I\}$. Given $(\xi, \tau) \in \mathbf{R}^{n+1}$ and for $(x, t) \in \mathbf{R}^{n+1}$ we let

$$P_0^{(\xi, \tau)} u(x, t) \quad \text{and} \quad P_1^{(\xi, \tau)} u(x, t)$$

denote the polynomials of δ_λ -degree $m \in \{0, 1\}$ such that (2.9) holds. Above (ξ, τ) is the point around which we approximate u , i.e. it corresponds to y in (2.9). Using this notation we prove:

Lemma 2.3 *Let $m \in \{0, 1\}$ and $\alpha \in (0, 1)$. Then $u \in \mathcal{C}_X^{m, \alpha}(D)$ if and only if there exists a constant c such that*

$$\left| u(x, t) - P_m^{(\xi, \tau)} u(x, t) \right| \leq c d_p((x, t), (\xi, \tau))^{m+\alpha} \quad (2.12)$$

whenever $(x, t), (\xi, \tau) \in D$.

Proof. If $u \in \mathcal{C}_X^{m, \alpha}(D)$ then by (2.10) we have that

$$P_0^{(\xi, \tau)} u(x, t) = u(\xi, \tau)$$

so (2.12) holds with $c = \|u\|_{\mathcal{C}_X^{0, \alpha}(D)}$ by the definition of $\mathcal{C}_X^{m, \alpha}(D)$. For $m = 1$

$$P_1^{(\xi, \tau)} u(x, t) = u(\xi, \tau) + \sum_{i=1}^q X_i u(\xi, \tau) \xi_i((\xi, t)^{-1} \circ (x, t)),$$

and since $\{X_i\}_{i=1}^q$ belong to the first layer of the stratification and also to the Jacobian basis $\zeta_i((\xi, t)^{-1} \circ (x, t)) = ((\xi, t)^{-1} \circ (x, t))_i = x_i - \xi_i$ (note that this is not the case for $i > q$). Once again (2.12) holds with $c = \|u\|_{\mathcal{C}_X^{1,\alpha}(D)}$ by the definition of $\mathcal{C}_X^{m,\alpha}(D)$.

Now assume that (2.12) holds. For $m = 0$ (2.12) reads

$$|u(x, t) - u(\xi, \tau)| \leq cd_p((x, t), (\xi, \tau))^\alpha, \quad (2.13)$$

and in particular $u \in \mathcal{C}_X^{0,\alpha}(D)$. For $m = 1$ (2.12) reads

$$\left| u(x, t) - u(\xi, \tau) + \sum_{i=1}^q X_i u(\xi, \tau)(x_i - \xi_i) \right| \leq cd_p((x, t), (\xi, \tau))^{1+\alpha}, \quad (2.14)$$

and by Lemma 2.1 $|X_i u(x, t) - X_i u(\xi, \tau)| \leq cd_p((x, t), (\xi, \tau))^\alpha$. Using this in the inequality (2.14) we find that there exists a constant c , depending on D , such that (2.13) holds and therefore it yields that $u \in \mathcal{C}_X^{1,\alpha}(D)$. \square

2.3 Function classes for the Cauchy-Dirichlet problem and the obstacle problem

In our proof of Theorem 1.1 and Theorem 1.2 we will make use of certain estimates, at the initial state, for the Cauchy-Dirichlet problem

$$\begin{cases} \mathcal{H}u(x, t) = f(x, t), & \text{in } D, \\ u(x, t) = g(x, t), & \text{on } \partial_P D, \end{cases} \quad (2.15)$$

where $D \subset \mathbf{R}^{n+1}$ is a bounded cylindrical domain in the sense of (2.7). In the following we introduce certain function classes using which we are able to give streamlined proofs of important lemmas and to prove Theorem 1.1 and Theorem 1.2 in a unified way. Function classes are introduced both for the Cauchy-Dirichlet problem and for the obstacle problem.

Definition 2.4 Let \mathcal{H} be defined as in (1.1), assume (1.3)-(1.6), let $D \subset \mathbf{R}^{n+1}$ be a bounded cylindrical domain in the sense of (2.7), let $m \in \{0, 1, 2\}$, $\alpha \in (0, 1)$, and let M_1, M_2, M_3 be three positive constants. Then we say that (u, f, g) belongs to the class $\mathcal{D}_m(\mathcal{H}, \alpha, D, M_1, M_2, M_3)$ if u is a solution to problem (2.15) with $f \in \mathcal{C}_X^{0,\alpha}(D)$, $g \in \mathcal{C}_X^{m,\alpha}(\overline{D})$ and

$$\|u\|_{L^\infty(D)} \leq M_1, \quad \|f\|_{\mathcal{C}_X^{0,\alpha}(D)} \leq M_2, \quad \|g\|_{\mathcal{C}_X^{m,\alpha}(D)} \leq M_3.$$

Definition 2.5 Let \mathcal{H} be defined as in (1.1), assume (1.3)-(1.6), let $D \subset \mathbf{R}^{n+1}$ be a bounded cylindrical domain in the sense of (2.7), let $m \in \{0, 1, 2\}$, $\alpha \in (0, 1)$, and let M_1, M_2, M_3 be three positive constants. Let $\varphi \in \mathcal{C}_X^{m,\alpha}(D)$, $g \in \mathcal{C}^0(\overline{D})$, $g \geq \varphi$ on $\partial_P D$, and let u be a strong solution to problem (1.2). Then, for $m \in \{0, 1, 2\}$ we say that (u, f, g, φ) belongs to the class $\mathcal{P}_m(\mathcal{H}, \alpha, D, M_1, M_2, M_3)$ if

$$\|u\|_{L^\infty(D)} \leq M_1, \quad \|f\|_{\mathcal{C}_X^{0,\alpha}(D)} \leq M_2, \quad \|\varphi\|_{\mathcal{C}_X^{m,\alpha}(D)} \leq M_3.$$

Definition 2.6 Let \mathcal{H} be defined as in (1.1), assume (1.3)-(1.6), let $D \subset \mathbf{R}^{n+1}$ be a bounded cylindrical domain in the sense of (2.7), let $m \in \{0, 1, 2\}$, $\alpha \in (0, 1)$, and let M_1, M_2, M_3, M_4 be four positive constants. Then, for $m \in \{0, 1, 2\}$ we say that (u, f, g, φ) belongs to the class $\tilde{\mathcal{P}}_m(\mathcal{H}, \alpha, D, M_1, M_2, M_3, M_4)$ if u is a strong solution to problem (1.2) with $f \in \mathcal{C}_X^{0,\alpha}(D)$, $\varphi, g \in \mathcal{C}_X^{m,\alpha}(\overline{D})$, $g \geq \varphi$ on $\partial_P D$ and

$$\|u\|_{L^\infty(D)} \leq M_1, \quad \|f\|_{\mathcal{C}_X^{0,\alpha}(D)} \leq M_2, \quad \|g\|_{\mathcal{C}_X^{m,\alpha}(D)} \leq M_3, \quad \|\varphi\|_{\mathcal{C}_X^{m,\alpha}(D)} \leq M_4.$$

As we will see below there are advantages, from the perspective of notation, to introduce both of the classes $\mathcal{P}_m(\mathcal{H}, \alpha, D, M_1, M_2, M_3)$ and $\tilde{\mathcal{P}}_m(\mathcal{H}, \alpha, D, M_1, M_2, M_3, M_4)$ though the classes are similar. In fact, we will use the class \mathcal{P}_m in the proof of Theorem 1.1 and the class $\tilde{\mathcal{P}}_m$ in the proof of Theorem 1.2. Furthermore, we note that our proofs of Theorem 1.1 and Theorem 1.2 will be based on certain blow-up arguments and in the following we lay out the fundamental notation concerning translations and dilations. The reason this can be done in an efficient way is due to the assumption concerning the existence of the homogeneous Lie group $\mathbf{G} = (\mathbf{R}^n, \circ, \delta_\lambda)$. In particular, let $D \subset \mathbf{R}^{n+1}$ be a bounded cylindrical domain in the sense of (2.7), assume that $(0, 0) \in D$, and let $v \in \mathcal{C}^0(D)$. We define the blow-up of a function v , at $(0, 0)$, as

$$v^r(x, t) = v^{r,(0,0)}(x, t) := v(\delta_r(x, t)), \quad r > 0,$$

whenever $\delta_r(x, t) \in D$. Recall that $\delta_r(x, t)$ was defined in Section 2.1. Using this notation a direct computation shows that

$$\mathcal{H}u = f \text{ in } D \quad \text{if and only if} \quad \mathcal{H}_r u^r = r^2 f^r \text{ in } \delta_{1/r}(D), \quad (2.16)$$

where

$$\mathcal{H}_r = \sum_{i,j=1}^q a_{ij}^r X_i^r X_j^r + \sum_{i=1}^q r b_i^r X_i^r - \partial_t, \quad (2.17)$$

since the vector fields $\{X_i\}_{i=1}^q$ are δ_λ -homogeneous of degree 1. In particular, $u^r(x, t) = u(\delta_r(x, t))$, $a_{ij}^r(x, t) = a_{ij}(\delta_r(x, t))$, $b_i^r(x, t) = b_i(\delta_r(x, t))$ whenever $\delta_r(x, t) \in D$. Furthermore, $X_i^r = (c_{i1}^r(x), \dots, c_{in}^r(x)) \cdot (\partial_{x_1}, \dots, \partial_{x_n})$, where $c_{ik}^r(x) = c_{ik}(\delta_r(x))$. Note that $X^r = \{X_1^r, \dots, X_m^r\}$ is still a system of smooth vector fields satisfying (1.3) and $A^r = \{a_{ij}^r\}$ and b_i^r satisfy, for $r \in (0, 1]$, (1.5) and (1.6) with the same constant as $A = \{a_{ij}\}$ and b_i . To proceed we also define, given $r \in (0, 1]$ and $(x_0, t_0) \in \mathbf{R}^{n+1}$,

$$u^{r,(x_0,t_0)}(x, t) = u((x_0, t_0) \circ \delta_r(x, t)). \quad (2.18)$$

Let $m \in \{0, 1, 2\}$, $\alpha \in (0, 1)$ and consider $r \in (0, 1]$. We then remark that $u \in \mathcal{C}_X^{m,\alpha}(D)$ if and only if $u^{r,(x_0,t_0)} \in \mathcal{C}_X^{m,\alpha}(\delta_{1/r}(D))$ and we note that

$$\|u^{r,(x_0,t_0)}\|_{\mathcal{C}_X^{m,\alpha}(\delta_{1/r}((x_0,t_0)^{-1} \circ D))} \leq \|u\|_{\mathcal{C}_X^{m,\alpha}(D)}. \quad (2.19)$$

Indeed in the case $m = 0$ we have

$$\|u^{r,(x_0,t_0)}\|_{\mathcal{C}_X^{m,\alpha}(\delta_{1/r}(D))} = \sup_D |u| + r^\alpha \sup_{\substack{z, \zeta \in D \\ z \neq \zeta}} \frac{|u(z) - u(\zeta)|}{d_p(z, \zeta)^\alpha} \leq \|u\|_{\mathcal{C}_X^{m,\alpha}(D)}.$$

Moreover

$$\mathcal{H}u = f \text{ in } (x_0, t_0) \circ \delta_r(D) \quad \text{if and only if} \quad \mathcal{H}_r^{(x_0,t_0)} u^{r,(x_0,t_0)} = r^2 f^{r,(x_0,t_0)} \text{ in } D,$$

where

$$\mathcal{H}_r^{(x_0,t_0)} = \sum_{i,j=1}^q a_{ij}^{r,(x_0,t_0)}(x, t) X_i^{r,(x_0,t_0)} X_j^{r,(x_0,t_0)} + \sum_{i=1}^q r b_i^{r,(x_0,t_0)}(x, t) X_i^{r,(x_0,t_0)} - \frac{\partial}{\partial t}.$$

Above $a_{ij}^{r,(x_0,t_0)}$, $b_i^{r,(x_0,t_0)}$ and $X_i^{r,(x_0,t_0)}$ are defined as in (2.18). In particular, we can conclude that if $(x_0, t_0) \in \mathbf{R}^{n+1}$, $r \in (0, 1]$, and if $(u, f, g, \varphi) \in \mathcal{P}_m(\mathcal{H}, \alpha, D, M_1, M_2, M_3)$, then

$$(u^{r,(x_0,t_0)}, f^{r,(x_0,t_0)}, g^{r,(x_0,t_0)}, \varphi^{r,(x_0,t_0)}) \in \mathcal{P}_m(\mathcal{H}_r^{(x_0,t_0)}, \alpha, (x_0, t_0) \circ \delta_r(D), \alpha, M_1, M_2, M_3).$$

The same statement holds for the class $\tilde{\mathcal{P}}_m(\mathcal{H}, \alpha, D, M_1, M_2, M_3, M_4)$.

3 Estimates for Parabolic Non-divergence Operators of Hörmander type

In this section we collect a number of results concerning parabolic non-divergence operators of Hörmander type. These results will be used in the proof of Theorem 1.1 and Theorem 1.2. In particular, let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). We first state some results from [BBLU09]. In particular, in [BBLU09] it is proved that there exists a fundamental solution, Γ , for \mathcal{H} on \mathbf{R}^{n+1} with a number of important properties: Γ is a continuous function away from the diagonal of $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ and $\Gamma(x, t, \xi, \tau) = 0$ for $t \leq \tau$. Moreover,

$$\Gamma(\cdot, \cdot, \xi, \tau) \in \mathcal{C}_{X,loc}^{2,\alpha}(\mathbf{R}^{n+1} \setminus \{(\xi, \tau)\}) \text{ for every fixed } (\xi, \tau) \in \mathbf{R}^{n+1}$$

and $\mathcal{H}(\Gamma(\cdot, \cdot, \xi, \tau)) = 0$ in $\mathbf{R}^{n+1} \setminus \{(\xi, \tau)\}$. For every $\psi \in C_0^\infty(\mathbf{R}^{n+1})$ the function

$$w(x, t) = \int_{\mathbf{R}^{n+1}} \Gamma(x, t, \xi, \tau) \psi(\xi, \tau) d\xi d\tau$$

belongs to $\mathcal{C}_{X,loc}^{2,\alpha}(\mathbf{R}^{n+1} \setminus \{(\xi, \tau)\})$ and $\mathcal{H}u = \psi$ in \mathbf{R}^{n+1} . Furthermore, let $\mu \geq 0$ and $T_2 > T_1$ be such that $(T_2 - T_1)\mu$ is small enough, let $g \in \mathcal{C}_X^{0,\beta}(\mathbf{R}^n \times [T_1, T_2])$, $0 < \beta \leq \alpha$, and let $f \in \mathcal{C}^0(\mathbf{R}^n)$ be such that $|g(x, t)|, |f(x)| \leq c \exp(\mu d(x, 0)^2)$ for some constant $c > 0$. Then the function

$$u(x, t) = \int_{\mathbf{R}^n} \Gamma(x, t, \xi, T_1) f(\xi) d\xi + \int_{T_1}^t \int_{\mathbf{R}^n} \Gamma(x, t, \xi, \tau) g(\xi, \tau) d\xi d\tau, \quad x \in \mathbf{R}^n, \quad t \in (T_1, T_2],$$

belongs to the class $\mathcal{C}_{X,loc}^{2,\alpha}(\mathbf{R}^n \times (T_1, T_2)) \cap \mathcal{C}^0(\mathbf{R}^n \times [T_1, T_2])$. Moreover, u solves the Cauchy-Dirichlet problem

$$\mathcal{H}u = g \text{ in } \mathbf{R}^n \times (T_1, T_2), \quad u(\cdot, T_1) = f(\cdot) \text{ in } \mathbf{R}^n.$$

Concerning the fundamental solution we will use the following uniform Gaussian bounds, see Theorem 10.7 in [BBLU09].

Lemma 3.1 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Then the fundamental solution Γ for \mathcal{H} on \mathbf{R}^{n+1} satisfies the following estimates. There exist a positive constant $C = C(\mathcal{H})$, and for every $T > 0$, a positive constant $c = c(\mathcal{H}, T)$ such that if $0 < t - \tau \leq T$ and $x, \xi \in \mathbf{R}^n$, then*

$$\begin{aligned} (i) & c^{-1} |B(x, \sqrt{t - \tau})|^{-1} e^{-Cd(x, \xi)^2/(t - \tau)} \leq \Gamma(x, t, \xi, \tau) \leq c |B(x, \sqrt{t - \tau})|^{-1} e^{-C^{-1}d(x, \xi)^2/(t - \tau)}, \\ (ii) & |X_i \Gamma(\cdot, t, \xi, \tau)(x)| \leq c(t - \tau)^{-1/2} |B(x, \sqrt{t - \tau})|^{-1} e^{-C^{-1}d(x, \xi)^2/(t - \tau)}, \\ (iii) & |X_i X_j \Gamma(\cdot, t, \xi, \tau)(x)| + |\partial_t \Gamma(x, \cdot, \xi, \tau)(t)| \leq c(t - \tau)^{-1} |B(x, \sqrt{t - \tau})|^{-1} e^{-C^{-1}d(x, \xi)^2/(t - \tau)}. \end{aligned}$$

Let Q be the homogeneous dimension of the underlying Lie group \mathbf{G} as introduced in (2.2). Using Q , (i) in Lemma 3.1 can be rewritten as

$$(i') \quad c^{-1} |t - \tau|^{-Q/2} e^{-Cd(x, \xi)^2/(t - \tau)} \leq \Gamma(x, t, \xi, \tau) \leq c |t - \tau|^{-Q/2} e^{-C^{-1}d(x, \xi)^2/(t - \tau)},$$

for $0 < t - \tau \leq T$. Recall the notion of cylinders in (1.8); in a similar fashion we now define

$$\begin{aligned} C_{r_1, r_2}(x, t) &= B_d(x, r_1) \times (t - r_2^2, t + r_2^2), \\ C_{r_1, r_2}^+(x, t) &= B_d(x, r_1) \times (t, t + r_2^2), \\ C_{r_1, r_2}^-(x, t) &= B_d(x, r_1) \times (t - r_2^2, t), \end{aligned}$$

whenever $(x, t) \in \mathbf{R}^{n+1}$ and $r_1, r_2 > 0$. The following Harnack inequality is proved in Theorem 15.1 in [BBLU09].

Lemma 3.2 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Let $R > 0$, $0 < h_1 < h_2 < 1$ and $\gamma \in (0, 1)$. Then there exists a positive constant $c = c(h_1, h_2, \gamma, R)$ such that the following holds for every $(\xi, \tau) \in \mathbf{R}^{n+1}$, $r \in (0, R]$. If*

$$u \in \mathcal{C}_X^{2,\alpha}(C_r^-(\xi, \tau)) \cap \mathcal{C}^0(\overline{C_r^-(\xi, \tau)})$$

satisfies $\mathcal{H}u = 0$, $u \geq 0$, in $C_r^-(\xi, \tau)$, then

$$u(x, t) \leq cu(\xi, \tau) \text{ whenever } (x, t) \in \overline{B_d(\xi, \gamma r)} \times [\tau - h_2 r^2, \tau - h_1 r^2].$$

In addition we will need the following Schauder estimate for the operator \mathcal{H} , which is proved in Theorem 1.1 in [BB07].

Lemma 3.3 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Let $R > 0$ and $(x, t) \in \mathbf{R}^{n+1}$. If $u \in \mathcal{C}_{X,loc}^{2,\alpha}(C_R(x, t))$ satisfies $\mathcal{H}u = f$ in $C_R(x, t)$, for some function $f \in \mathcal{C}_X^{0,\alpha}(C_R(x, t))$, then there exists a positive constant c , depending on \mathcal{H} , α and R such that*

$$\|u\|_{\mathcal{C}_X^{2,\alpha}(C_{R/2}(x, t))} \leq c \left(\|u\|_{L^\infty(C_R(x, t))} + \|f\|_{\mathcal{C}_X^{0,\alpha}(C_R(x, t))} \right).$$

We will also need the weak maximum principle for the operator \mathcal{H} and we will use one due to Picone, e.g., see Theorem 13.1 in [BBLU09].

Lemma 3.4 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Let $D \subset \mathbf{R}^{n+1}$ be a bounded cylindrical domain in the sense of (2.7). Then the following holds; if $u \in \mathcal{C}_X^{2,\alpha}(D)$, $\mathcal{H}u \geq 0$ in D and $\limsup u \leq 0$ on $\partial_p D$, then $u \leq 0$ in D .*

Finally, we have to ensure that the continuous Cauchy-Dirichlet problem is solvable in the cylinders $C_r(x, t)$, $C_r^+(x, t)$, $C_r^-(x, t)$ introduced in (1.8). However, as previously mentioned, this is not the case in general and we will have to work with modified cylinders. To explain this further, in [Ugu07], Uguzzoni develops what he refers to as a "cone criterion" for the solvability of the Cauchy-Dirichlet problem for non-divergence operators structured on Hörmander vector fields. This is a generalization of the well-known positive density condition in classical potential theory. In the following we describe his results in the setting of cylindrical domains $D = \Omega \times (T_1, T_2)$, where Ω is an open, bounded domain in \mathbf{R}^n . A bounded, open set Ω is said to have outer positive d -density at $x_0 \in \partial\Omega$ if there exist $r_0, \theta > 0$ such that

$$|B_d(x_0, r) \setminus \overline{\Omega}| \geq \theta |B_d(x_0, r)|, \quad \text{for all } r \in (0, r_0),$$

where $|\cdot|$ denotes Euclidean volume. Furthermore, if Ω has outer positive d -density at all points $x \in \partial\Omega$ and if r_0 and θ can be chosen independent of x , then Ω is said to satisfy the uniform outer positive d -density condition. The following lemma is a special case of Lemma 4.1 in [Ugu07].

Lemma 3.5 *Let \mathcal{H} be defined as in (1.1), assume (1.3)-(1.6) and that f and g are continuous and bounded. Let $D \subset \mathbf{R}^{n+1}$ be a bounded cylindrical domain of the specific form $D = \Omega \times (T_1, T_2)$. Assume that Ω satisfies the uniform outer positive d -density condition. Then there exists a unique solution $u \in \mathcal{C}_X^{2,\alpha}(D) \cap \mathcal{C}^0(\overline{D})$ to the problem*

$$\mathcal{H}u = f \text{ in } D, \quad u = g \text{ on } \partial_p D,$$

where α is the Hölder exponent in (1.6).

The above lemma only applies for bounded domains satisfying the outer positive d -density condition and in general the cylinders $C_r(x, t)$, $C_r^+(x, t)$, $C_r^-(x, t)$ do not satisfy this criterion. However, in Theorem 6.5 in [LU10], Lanconelli and Uguzzoni prove that we can approximate these cylinders with modified versions which are regular for the Cauchy-Dirichlet problem.

Theorem 3.6 *Let Ω be a bounded domain in \mathbf{R}^n . Then, for every $\delta > 0$ there exist Ω_δ such that $\{x \in \Omega : d(x, \partial\Omega) > \delta\} \subset \Omega_\delta \subset \Omega$ and Ω_δ satisfies the uniform outer positive d -density condition.*

Remark 3.7 *Below, we will assume that we work with modified cylinders which are regular for the Cauchy-Dirichlet problem.*

3.1 Auxiliary technical estimates

In this subsection we derive a few auxiliary technical estimates to be used in the forthcoming sections.

Lemma 3.8 *Let \mathcal{H} be defined as in (1.1) with corresponding fundamental solution Γ and assume (1.3)-(1.6). Given $\gamma, R > 0$, we define the function*

$$u(x, t) = \int_{B_d(0, R)} \Gamma(x, t, y, 0) \|y\|^\gamma dy, \quad \text{for } x \in \mathbf{R}^n, \quad t > 0.$$

Let $D = \Omega \times (0, T)$ where $\Omega \subset \mathbf{R}^n$ is a bounded domain and let $T > 0$. Then there exists a positive constant $c = c(\gamma, T, \mathcal{H})$ such that

$$u(x, t) \leq c \|(x, t)\|_p^\gamma, \quad \text{for } (x, t) \in D.$$

Proof. First of all we note that

$$\|y\| = \|(y, 0)\|_p = \|(y, 0)^{-1}\|_p \leq \|(y, 0)^{-1} \circ (x, t)\|_p + \|(x, t)\|_p$$

by the triangle inequality. Hence,

$$\begin{aligned} u(x, t) &= \int_{B_d(0, R)} \Gamma(x, t, y, 0) \|y\|^\gamma dy \\ &\leq c(\gamma) \left(\| (x, t) \|_p^\gamma \int_{B_d(0, R)} \Gamma(x, t, y, 0) dy + \int_{B_d(0, R)} \Gamma(x, t, y, 0) \| (y, 0)^{-1} \circ (x, t) \|_p^\gamma dy \right). \end{aligned} \quad (3.1)$$

Below we will use Lemma 3.1 and the following estimates, see Proposition 10.11 and Corollary 10.12 in [BBLU09]: For any $\mu \geq 0$ there exist $c(\mu)$ such that

$$(d(x, y)^2/t)^\mu |B(x, \sqrt{\lambda t})|^{-1} \exp\left(-\frac{d(x, y)^2}{\lambda t}\right) \leq c(\mu, \mathcal{H}) \lambda^\mu |B(x, \sqrt{2\lambda t})|^{-1} \exp\left(-\frac{d(x, y)^2}{2\lambda t}\right)$$

for every $\lambda > 0$, $x, y \in \mathbf{R}^n$, $t > 0$, and

$$\int_{\mathbf{R}^n} |B(x, \sqrt{ct})|^{-1} \exp\left(-\frac{d(x, y)^2}{ct}\right) dy \leq c(T, \mathcal{H})$$

for $0 < t \leq T$, $x \in \mathbf{R}^n$. Continuing, using the above estimates, the first integral in (3.1) is bounded by some constant depending on T while the second integral is bounded by

$$\begin{aligned} &\int_{B_d(0, R)} \Gamma(x, t, y, 0) \| (y, 0)^{-1} \circ (x, t) \|_p^\gamma dy \\ &\leq \int_{B_d(0, R)} |B(x, \sqrt{ct})|^{-1} \exp\left(-\frac{d(x, y)^2}{ct}\right) (d(x, y)^2 + t)^{\gamma/2} dy \\ &\leq c(\gamma, \mathcal{H}) t^{\gamma/2} \left(\int_{B_d(0, R)} |B(x, \sqrt{ct})|^{-1} \exp\left(-\frac{d(x, y)^2}{ct}\right) (d(x, y)^2/t)^{\gamma/2} dy + \right. \\ &\quad \left. \int_{B_d(0, R)} |B(x, \sqrt{ct})|^{-1} \exp\left(-\frac{d(x, y)^2}{ct}\right) dy \right) \\ &\leq c(\gamma, T, \mathcal{H}) t^{\gamma/2} \leq c(\gamma, T, \mathcal{H}) \| (x, t) \|_p^\gamma. \end{aligned}$$

The dependency on \mathcal{H} stems from the constants in Lemma 3.1. This concludes the proof. \square

Lemma 3.9 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Let Q be the homogeneous dimension of the underlying Lie group \mathbf{G} as introduced in (2.2). Let $K \gg 1$ and R be given. Assume that $\mathcal{H}u = 0$ in $C_{KR, R}^-(0, 0)$ and that $u(x, -R^2) = 0$ whenever $x \in \overline{B_d(0, KR)}$. Then there exists a constant $c = c(\mathcal{H}, T)$ such that*

$$\sup_{C_R^-} |u| \leq cK^{Q+1} e^{-cK^2} \sup_{\partial_p C_{KR, R}^-(0, 0) \cap \{(x, t) : t > -R^2\}} |u|,$$

Proof. Let $r > 0$ be small enough so that $\{y : \|y\| < 2r\} \subset B_d(0, 1)$ and let $\phi \in C^\infty(\mathbf{R}^n)$ be a function taking values in $[0, 1]$ such that

$$\begin{cases} \phi(x) = 1, & \text{if } \|x\| \geq 2r, \\ \phi(x) = 0, & \text{if } \|x\| \leq r. \end{cases}$$

We define

$$\omega(x, t) := 2 \int_{\mathbf{R}^n} \Gamma(x, t, y, -R^2) \phi(\delta_{1/KR}(y)) dy,$$

and we note that ω is a non-negative solution to the Cauchy-problem

$$\begin{cases} \mathcal{H}u = 0, & \text{in } \mathbf{R}^n \times (-R^2, 0], \\ u(x, -R^2) = 2\phi(\delta_{1/KR}(x)), & \text{on } \mathbf{R}^n. \end{cases}$$

Furthermore, we note that for $(x, t) \in \partial_p C_{KR}^-(0, 0) \cap \{(x, t) : t > -R^2\}$ the dilated point $\delta_{1/KR}(x, t)$ belongs to $C_{1,1/K^2}^-(0, 0)$ and that, by Lemma 3.1,

$$\omega(x, t) \geq 2c^{-1} \int_{\mathbf{R}^n} \left| B\left(x, \sqrt{t+R^2}\right) \right|^{-1} \exp\left(-C \frac{d(x, y)^2}{t+R^2}\right) \phi(\delta_{1/KR}(y)) dy.$$

We intend to prove that the integral above converges to 2 as $K \rightarrow \infty$ uniformly in $(x, t) \in \partial_p C_{KR}^-(0, 0) \cap \{(x, t) : t > -R^2\}$. Since, in that case, there exists a constant K_0 such that, for $K > K_0$, we have that $\omega(x, t) \geq 1$ for all $(x, t) \in \partial_p C_{KR}^-(0, 0) \cap \{(x, t) : t > -R^2\}$. Then using the assumptions the weak maximum principle implies that

$$u(x, t) \leq \omega(x, t) \sup_{\partial_p C_{KR,R}^-(0,0) \cap \{(x,t): t > -R^2\}} |u|,$$

for boundary points and we are done if we, in addition, can prove that $\omega(x, t) \leq cK^{Q+1}e^{-cK^2}$ for $(x, t) \in C_R^-(0, 0) \cap \{(x, t) : t > -R^2\}$.

Now,

$$\begin{aligned} |\omega(x, t) - 1| &\leq c^{-1} \int_{\mathbf{R}^n} \left| B\left(x, \sqrt{t+R^2}\right) \right|^{-1} \exp\left(-C \frac{d(x, y)^2}{t+R^2}\right) |\phi(\delta_{1/KR}(y)) - 1| dy \\ &\leq c^{-1} \int_{B_d(0, 2rKR)} \left| B\left(x, \sqrt{t+R^2}\right) \right|^{-1} \exp\left(-C \frac{d(x, y)^2}{t+R^2}\right) dy \\ &\leq c^{-1} R \int_{B_d(0, 2rKR)} \left(\sqrt{C} \frac{d(x, y)}{\sqrt{t+R^2}} \right)^{Q+1} \exp\left(-C \frac{d(x, y)^2}{t+R^2}\right) \left(\frac{1}{\sqrt{C}d(x, y)} \right)^{Q+1} dy \\ &\leq c^{-1} Rc(Q, \mathcal{H}) \int_{B_d(0, 2rKR)} d(x, y)^{-Q-1} dy \\ &\leq c^{-1} Rc(Q, \mathcal{H})(2rKR)^Q \frac{1}{\left(\left(\frac{1}{c_d} - 2r\right)KR\right)^{Q+1}} \\ &= c(\Omega, Q_T, \mathcal{H}) \frac{1}{K} \rightarrow 0 \end{aligned}$$

as $K \rightarrow \infty$ for $(x, t) \in C_{KR,R}^-(0, 0) \cap \{(x, t) : t > -R^2\}$. Above we used that $2r < 1$. It remains to show that $\omega(x, t) \leq cK^{Q+1}e^{-cK^2}$ for $(x, t) \in C_R^-(0, 0) \cap \{(x, t) : t > -R^2\}$. To prove this, recall that $u^n e^{-u^2}$ is bounded. A direct calculation then shows that

$$\begin{aligned} \omega(x, t) &\leq c \int_{\|y\| \geq rKR} \left| B\left(x, \sqrt{t+R^2}\right) \right|^{-1} \exp\left(-C \frac{d(x, y)^2}{t+R^2}\right) dy \\ &\leq c \int_{\|y\| \geq rKR} (t+R^2)^{-(Q+1)/2} \exp\left(-\frac{C}{4} \frac{d(0, y)^2}{(t+R^2)}\right) dy \\ &\leq c \int_{\|y\| \geq rKR} \left(\frac{C}{8} \frac{d(0, y)^2}{K^2(t+R^2)} \right)^{(Q+1)/2} \exp\left(-\frac{C}{8} \frac{d(0, y)^2}{K^2(t+R^2)}\right) \\ &\quad \cdot \exp\left(-\frac{C}{8} \frac{d(0, y)^2}{t+R^2}\right) \left(\frac{8}{C} \frac{K^2}{d(0, y)^2} \right)^{(Q+1)/2} dy \\ &\leq cK^{Q+1} \int_{\|y\| \geq rKR} d(0, y)^{-(Q+1)} \exp\left(-\frac{C}{8} \frac{d(0, y)^2}{t+R^2}\right) dy \\ &\leq cK^{Q+1} e^{-cK^2} \int_{\|y\| \geq rKR} d(0, y)^{-(Q+1)} dy. \end{aligned} \tag{3.2}$$

Now, we define $S_j := \{y : 2^j rKR \leq \|y\| \leq 2^{j+1} rKR\}$ for $j \in \{0, 1, \dots\}$, and we use (3.2) together with (2.6), to obtain

$$\begin{aligned} \omega(x, t) &\leq c(\mathcal{H}, T) K^{Q+1} e^{-CK^2} \sum_{j=0}^{\infty} (2^j rKR)^Q (2^j rKR)^{-(Q+1)} \\ &\leq c(\mathcal{H}, T) K^{Q+1} e^{-CK^2} \sum_{j=0}^{\infty} 2^{-j} \leq C(\mathcal{H}, R_0) K^{Q+1} e^{-CK^2}, \end{aligned}$$

which completes the proof of the lemma. It should be clear that the constant only depend on \mathcal{H} and T , and that this dependency originates in the use of Lemma 3.1. \square

Lemma 3.10 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Let $\alpha \in (0, 1)$, $m = 0, 1, 2$ and let M_1, M_2, M_3 be positive constants. Assume that*

$$(u, f, g) \in \mathcal{D}_m(\mathcal{H}, \alpha, C^+, M_1, M_2, M_3).$$

Then there exists a constant $c = c(\mathcal{H}, m, \alpha, M_1, M_2, M_3)$ such that

$$\sup_{C_r^+} |u - g| \leq cr^\gamma, \quad r \in (0, 1),$$

where $\gamma = m + \alpha$ for $m = 0, 1$ and $\gamma = 2$ for $m = 2$.

Proof. Recall that $C^+ = C_1^+(0, 0)$ and that $C_r^+ = C_r^+(0, 0)$. To start the proof we first note, using the triangle inequality and Lemma 2.3, that it suffices to prove that

$$\sup_{C_r^+} |u - P_m^{(0,0)} g| \leq cr^\gamma, \quad r \in (0, 1),$$

where $\gamma = m + \alpha$ for $m = 0, 1$ and that

$$\sup_{C_r^+} |u - g| \leq cr^\gamma, \quad r \in (0, 1),$$

where $\gamma = 2$ for $m = 2$. Consider $m = 0, 1$ and let $v_m = u - P_m^{(0,0)} g$. Then v_m satisfies the equation

$$\mathcal{H}v_m = f - \mathcal{H}P_m^{(0,0)} g =: f_m.$$

Since $f_m \in C_X^{0,\alpha}$ we see that we can assume in case $m = 0, 1$, without loss of generality, that $P_m^{(0,0)} g = 0$. By subtraction of g we can, using the same argument, assume that $g \equiv 0$ in case $m = 2$. After these preliminary problem reductions we proceed with the proof and we first consider the case $m = 0$. Let

$$\partial_p^+ C^+ = \partial_p C^+ \cap \{t > 0\}, \quad \partial_p^- C^+ = \partial_p C^+ \cap \{t = 0\}.$$

Let v_1, v_2 and v_3 respectively be the unique solutions of the following boundary value problems,

$$\begin{cases} \mathcal{H}v_1 = 0 & \text{in } C^+, \\ v_1 = 0 & \text{on } \partial_p^+ C^+, \\ v_1 = g & \text{on } \partial_p^- C^+, \end{cases} \quad \begin{cases} \mathcal{H}v_2 = 0 & \text{in } C^+, \\ v_2 = g & \text{on } \partial_p^+ C^+, \\ v_2 = 0 & \text{on } \partial_p^- C^+, \end{cases} \quad \begin{cases} \mathcal{H}v_3 = -\|f\|_{C_X^{0,\alpha}(C^+)} & \text{in } C^+, \\ v_3 = 0 & \text{on } \partial_p C^+. \end{cases}$$

Note that using Lemma 3.5, Theorem 3.6 and Remark 3.7 we see that C^+ is regular for the Cauchy-Dirichlet problem for \mathcal{H} and that v_1, v_2 and v_3 are well-defined. Then, by the maximum principle, see Lemma 3.4, we have

$$v_1 + v_2 - v_3 \leq u \leq v_1 + v_2 + v_3 \quad \text{in } C^+,$$

so that we only have to prove, since $\gamma = \alpha$ in the case $m = 0$, that

$$\sup_{C_r^+} (|v_1| + |v_2| + |v_3|) \leq cr^\alpha, \tag{3.3}$$

for r suitably small. To proceed, since $\|g\|_{C_X^{0,\alpha}(C^+)} \leq M_3$ we see by the maximum principle that

$$|v_1(x, t)| \leq M_3 \int_{\mathbf{R}^n} \Gamma(x, t, y, 0) \|y\|^\alpha dy$$

and hence we can apply Lemma 3.8 to conclude that

$$|v_1(x, t)| \leq cM_3 \|(x, t)\|_p^\alpha.$$

We next apply Lemma 3.9 to conclude that

$$\sup_{C_r^+} |v_2| \leq cr^Q \exp(-cr^{-2}) \sup_{\partial_p^+ C^+} |v_2|$$

for any $r \leq K^{-1}$. Since $|v_2|$ agrees with $|u|$ on $\partial_p^+ C^+$ we can conclude that

$$\sup_{C_r^+} |v_2| \leq cM_1 r^Q \exp(-c_1 r^{-2}) \leq c_2 M_1 r^2, \quad \text{for every } r \in (0, K^{-1}).$$

Finally, we have

$$|v_3(x, t)| \leq \|f\|_{C_X^{0,\alpha}(C^+)} \int_0^t \int_{\mathbf{R}^n} \Gamma(x, t, y, s) dy ds \leq ct \|f\|_{C_X^{0,\alpha}(C^+)} \leq cM_2 \|(x, t)\|_p^2.$$

This proves (3.3) and the proof of the lemma is complete in the case $m = 0$. The cases $m = 1, 2$ can be handled by repeating the argument above. In particular, we now apply Lemma 3.8 with $\gamma = m + \alpha$ for $m = 1$ and $\gamma = 2$ for $m = 2$ and we find

$$|v_1(x, t)| \leq cM_3 \|(x, t)\|_p^\gamma.$$

Using this estimate we are then able to complete the argument as above. This completes the proof. \square

4 Estimates for the obstacle problem

The purpose of this section is to derive certain estimates for the obstacle problem to be used in the proofs of Theorem 1.1 and Theorem 1.2. The results of the section are the following lemmas.

Lemma 4.1 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Let $\alpha \in (0, 1)$, $m = 0, 1, 2$, and let M_1, M_2, M_3 be positive constants. Assume that*

$$(u, f, g, \varphi) \in \mathcal{P}_m(\mathcal{H}, \alpha, C^-, M_1, M_2, M_3) \text{ and that } u(0, 0) = \varphi(0, 0).$$

Then there exists $c = c(\mathcal{H}, m, \alpha, M_1, M_2, M_3)$ such that

$$\sup_{C_r^-} |u - \varphi| \leq cr^\gamma, \quad r \in (0, 1),$$

where $\gamma = m + \alpha$ for $m = 0, 1$ and $\gamma = 2$ for $m = 2$.

Lemma 4.2 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Let $\alpha \in (0, 1)$, $m = 0, 1, 2$, and let M_1, M_2, M_3, M_4 be positive constants. Assume that*

$$(u, f, g, \varphi) \in \tilde{\mathcal{P}}_m(\mathcal{H}, \alpha, C^+, M_1, M_2, M_3, M_4).$$

Then there exists $c = c(\mathcal{H}, m, \alpha, M_1, M_2, M_3, M_4)$ such that

$$\sup_{C_r^+} |u - g| \leq cr^\gamma, \quad r \in (0, 1),$$

where $\gamma = m + \alpha$ for $m = 0, 1$ and $\gamma = 2$ for $m = 2$.

Lemma 4.3 *Let \mathcal{H} be defined as in (1.1) and assume (1.3)-(1.6). Let $\alpha \in (0, 1)$, $m = 0, 1, 2$, and let M_1, M_2, M_3 be positive constants. Assume that*

$$(u, f, g, \varphi) \in \mathcal{P}_m(\mathcal{H}, \alpha, C, M_1, M_2, M_3) \text{ and that } u(0, 0) = \varphi(0, 0).$$

Then there exists $c = c(\mathcal{H}, m, \alpha, M_1, M_2, M_3)$ such that

$$\sup_{C_r^-} |u - \varphi| \leq cr^\gamma, \quad r \in (0, 1),$$

where $\gamma = m + \alpha$ for $m = 0, 1$ and $\gamma = 2$ for $m = 2$.

4.1 Proof of Lemma 4.1

We first note, as in the proof of Lemma 3.10, that using the triangle inequality and Lemma 2.3 it suffices to prove that

$$\sup_{C_r^-} |u - P_m^{(0,0)} \varphi| \leq cr^\gamma, \quad r \in (0, 1), \quad (4.1)$$

where $\gamma = m + \alpha$ for $m = 0, 1$ and

$$\sup_{C_r^-} |u - \varphi| \leq cr^\gamma, \quad r \in (0, 1), \quad (4.2)$$

where $\gamma = 2$ for $m = 2$. Furthermore, by a simple problem reduction we see that we can assume, without loss of generality, that $P_m^{(0,0)} \varphi \equiv 0$ for $m = 0, 1$ and that $\varphi \equiv 0$ for $m = 2$. After these simplifications we define $S_k^-(u)$ as in (1.9) and we intend to prove that there exists a positive constant $\tilde{c} = \tilde{c}(\mathcal{H}, m, \alpha, M_1, M_2, M_3)$ such that (1.12) holds for all $k \in \mathbf{N}$ with F defined as in (1.11). Indeed, if (1.12) holds then we see, by a simple iteration argument, that

$$S_k^-(u - F) \leq \frac{\tilde{c}}{2^{k\gamma}}$$

and hence (4.1) and (4.2), and thereby Lemma 4.1, follows.

We first consider the case when $m = 0$ and prove (1.12) with $\gamma = \alpha$. In particular, we assume that

$$(u, f, g, \varphi) \in \mathcal{P}_0(\mathcal{H}, \alpha, C^-, M_1, M_2, M_3).$$

As in [FNPP10] we divide the argument into three steps.

Step 1. Using that $\varphi(0, 0) = 0$ by the problem reduction, we first note that

$$u(x, t) \geq \varphi(x, t) = \varphi(x, t) - \varphi(0, 0) \geq -M_3 \|(x, t)\|_p^\gamma, \quad (x, t) \in C^-. \quad (4.3)$$

Assume that (1.12) is false. Then for every $j \in \mathbf{N}$, there exists a positive integer k_j and functions $(u_j, f_j, g_j, \varphi_j) \in \mathcal{P}_0(\mathcal{H}, \alpha, C^-, M_1, M_2, M_3)$ such that $u_j(0, 0) = \varphi_j(0, 0) = 0$ and

$$S_{k_j+1}^-(u_j) > \max \left(\frac{jM_3}{2^{(k_j+1)\gamma}}, \frac{S_{k_j}^-(u_j)}{2^\gamma}, \frac{S_{k_j-1}^-(u_j)}{2^{2\gamma}}, \dots, \frac{S_0^-(u_j)}{2^{(k_j+1)\gamma}} \right), \quad (4.4)$$

while (1.12) is true for $k < k_j$. Using the definition of $S_{k_j+1}^-$ in (1.9) we see that there exists (x_j, t_j) in the closure of $C_{2^{-(k_j+1)}}^-$ such that $|u_j(x_j, t_j)| = S_{k_j+1}^-(u_j)$ for every $j \geq 1$. Moreover from (4.3) and (4.4) it follows that $u_j(x_j, t_j) > 0$. Using (4.4) we can conclude, as $|u_j| \leq M_1$, that $j2^{-\gamma k_j}$ is bounded and hence that $k_j \rightarrow \infty$ as $j \rightarrow \infty$.

Step 2. We define $(\tilde{x}_j, \tilde{t}_j) = \delta_{2^{k_j}}(x_j, t_j)$ and $\tilde{u}_j : C_{2^{k_j}}^- \rightarrow \mathbf{R}$,

$$\tilde{u}_j(x, t) = \frac{u_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^-(u_j)}.$$

In particular, we note that $(\tilde{x}_j, \tilde{t}_j)$ belongs to the closure of $C_{1/2}^-$ and

$$\tilde{u}_j(\tilde{x}_j, \tilde{t}_j) = 1. \quad (4.5)$$

Moreover we let $\tilde{\mathcal{H}}_j = \mathcal{H}_{2^{-k_j}}$, see (2.17) for the exact definition of the scaled operator, and let

$$\tilde{f}_j(x, t) = 2^{-2k_j} \frac{f_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^-(u_j)}, \quad \tilde{g}_j(x, t) = \frac{g_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^-(u_j)}, \quad \tilde{\varphi}_j(x, t) = \frac{\varphi_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^-(u_j)}, \quad (4.6)$$

whenever $(x, t) \in C_{2^{k_j}}^-$. Then, using (2.16) we see that \tilde{u}_j solves

$$\begin{cases} \max\{\tilde{\mathcal{H}}_j \tilde{u}_j - \tilde{f}_j, \tilde{\varphi}_j - \tilde{u}_j\} = 0, & \text{in } C_{2^{k_j}}^-, \\ \tilde{u}_j = \tilde{g}_j, & \text{on } \partial_p C_{2^{k_j}}^-. \end{cases}$$

Moreover, for any $l \in \mathbb{N}$ we have that

$$\sup_{C_{2^l}^-} |\tilde{u}_j| = \frac{S_{k_j-l}^-(u_j)}{S_{k_j+1}^-(u_j)} \leq 2^{(l+1)\gamma} \text{ whenever } k_j > l. \quad (4.7)$$

In particular, we can conclude that

$$(\tilde{u}_j, \tilde{f}_j, \tilde{u}_j, \tilde{\varphi}_j) \in \mathcal{P}_0(\tilde{\mathcal{H}}_j, \alpha, C_{2^l}^-, \tilde{M}_1^j, \tilde{M}_2^j, \tilde{M}_3^j), \text{ and that } \tilde{u}_j(0, 0) = \tilde{\varphi}_j(0, 0), \quad (4.8)$$

and, using (4.6) and (4.7), we have that

$$\tilde{M}_1^j \leq 2^{(l+1)\gamma}, \quad \tilde{M}_2^j \leq 2^{-2k_j} \frac{M_2}{S_{k_j+1}^-(u_j)}, \quad \tilde{M}_3^j \leq 2^{\gamma(l-k_j)} \frac{M_3}{S_{k_j+1}^-(u_j)}.$$

Now, by (4.4) we see that in $C_{2^l}^-$

$$\lim_{j \rightarrow \infty} \tilde{M}_2^j = \lim_{j \rightarrow \infty} \tilde{M}_3^j = 0. \quad (4.9)$$

Step 3. In the following we let l be a suitably large positive integer to be specified later. We consider $j_0 \in \mathbb{N}$ such that $k_j > 2^l$ for $j \geq j_0$. We let \hat{g}_j denote the boundary values of \tilde{u}_j on $\partial_p C_{2^l}^-$ and we let v_j and \tilde{v}_j be such that

$$\begin{cases} \tilde{\mathcal{H}}_j v_j = \|\tilde{f}_j\|_{L^\infty(C_{2^l}^-)} & \text{in } C_{2^l}^-, \\ v_j = \hat{g}_j & \text{on } \partial_p C_{2^l}^-, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\mathcal{H}}_j \tilde{v}_j = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^-)} & \text{in } C_{2^l}^-, \\ \tilde{v}_j = \max\{\hat{g}_j, \tilde{M}_3^j\} & \text{on } \partial_p C_{2^l}^-. \end{cases}$$

We shall prove that

$$v_j \leq \tilde{u}_j \leq \tilde{v}_j \text{ in } C_{2^l}^-. \quad (4.10)$$

The first inequality in (4.10) follows from the comparison principle. To prove the second one, we note that $\|\tilde{\varphi}_j\|_{L^\infty} \leq \tilde{M}_3^j$ and then by the maximum principle $\tilde{v}_j \geq \tilde{\varphi}_j$ in $C_{2^l}^-$. Furthermore,

$$\tilde{\mathcal{H}}_j(\tilde{v}_j - \tilde{u}_j) = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^-)} + \tilde{f}_j \leq 0 \quad \text{in } D := C_{2^l}^- \cap \{(x, t) : \tilde{u}_j(x, t) > \tilde{\varphi}_j(x, t)\},$$

and $\tilde{v}_j \geq \tilde{u}_j$ on $\partial_p D$. Hence, the second inequality in (4.10) follows from the maximum principle. Further, since $\tilde{u}_j \geq \tilde{\varphi}_j$ by (4.8), we can conclude that $\hat{g}_j \geq -\tilde{M}_3^j$ in $C_{2^l}^-$. Now we use the maximum principle to see that

$$\tilde{v}_j(x, t) - v_j(x, t) \leq \left(\max\{0, \tilde{M}_3^j - \hat{g}_j\} + 2\|\tilde{f}_j\|_{L^\infty(C_{2^l}^-)} \right) \leq 2 \left(\tilde{M}_3^j + \tilde{M}_2^j \right) \quad (4.11)$$

whenever $(x, t) \in C_{2^l, 1}^-$. We claim that there exists a positive constant c_1 such that

$$\tilde{v}_j(x, t) \geq c_1 \quad \text{for every } (x, t) \in C_{1/2}^-, \quad j \geq j_0. \quad (4.12)$$

Assuming (4.12) we use (4.11) and (4.10) to conclude that

$$\tilde{u}_j(0, 0) \geq v_j(0, 0) \geq \tilde{v}_j(0, 0) - 2 \left(\tilde{M}_3^j + \tilde{M}_2^j \right) \geq c_1 - 2 \left(\tilde{M}_3^j + \tilde{M}_2^j \right),$$

and then, by (4.9), that $\tilde{u}_j(0, 0) > 0$ for j suitably large. This contradicts the assumption that $\tilde{u}_j(0, 0) = \tilde{\varphi}_j(0, 0) = 0$. Hence our original assumption was incorrect and the proof of the lemma is complete. Based on

the above it only remains to prove (4.12). Our proof of (4.12) is based on Lemma 3.2 and Lemma 3.9. Below we will use the following short notation for $R_1, R_2 \geq 1$,

$$\partial_p^+ C_{R_1, R_2}^- = \partial_p C_{R_1, R_2}^- \cap \{t > -R_2^2\}, \quad \partial_p^- C_{R_1, R_2}^- = \partial_p C_{R_1, R_2}^- \cap \{t = R_2^2\},$$

where $C_{R_1, R_2}^- = B_d(0, R_1) \times (-R_2^2, 0]$. We write $\tilde{v}_j = w_j + \tilde{w}_j + \hat{w}_j$ on $C_{2^l, 1}^-$ where

$$\begin{cases} \tilde{\mathcal{H}}_j w_j = 0 & \text{in } C_{2^l, 1}^-, \\ w_j = 0 & \text{on } \partial_p^+ C_{2^l, 1}^-, \\ w_j = \tilde{v}_j & \text{on } \partial_p^- C_{2^l, 1}^-, \end{cases} \quad \begin{cases} \tilde{\mathcal{H}}_j \tilde{w}_j = 0 & \text{in } C_{2^l, 1}^-, \\ \tilde{w}_j = \tilde{v}_j & \text{on } \partial_p^+ C_{2^l, 1}^-, \\ \tilde{w}_j = 0 & \text{on } \partial_p^- C_{2^l, 1}^-, \end{cases} \quad \begin{cases} \tilde{\mathcal{H}}_j \hat{w}_j = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^-)} & \text{in } C_{2^l, 1}^-, \\ \hat{w}_j = 0 & \text{on } \partial_p C_{2^l, 1}^-. \end{cases}$$

By the maximum principle we see that

$$0 \leq \hat{w}_j(x, t) \leq (t+1)\|\tilde{f}_j\|_{L^\infty(C_{2^l}^-)} \leq \tilde{M}_2^j, \quad (4.13)$$

whenever $(x, t) \in C_{2^l, 1}^-$. Hence, as $t \in (-1, 0)$ we see that $|\hat{w}_j(x, t)| \leq 1/4$ in $C_{2^l, 1}^-$ if j is sufficiently large. Since

$$\|\tilde{v}_j\|_{L^\infty(C_{2^l}^-)} \leq \max\left\{2^{(l+1)\gamma}, \tilde{M}_3^j\right\} + 2^{2l}\tilde{M}_2^j, \quad (4.14)$$

by Lemma 3.9 we find

$$\sup_{C_{1/2}^-} |\tilde{w}_j| \leq c2^{(Q+1)l}e^{-c2^{2l}} \sup_{\partial_p^+ C_{2^l}^-} |v| \leq c2^{(Q+1)l}e^{-c2^{2l}} \left(\max\left\{2^{(l+1)\gamma}, \tilde{M}_3^j\right\} + 2^{2l}\tilde{M}_2^j\right). \quad (4.15)$$

In particular we note that the right hand side in this inequality tends to zero as l goes to infinity. Recalling that $\tilde{v}_j(\tilde{x}_j, \tilde{t}_j) \geq \tilde{u}_j(\tilde{x}_j, \tilde{t}_j) = 1$, we can conclude, by choosing l suitably large, that

$$w_j(\tilde{x}_j, \tilde{t}_j) \geq \frac{1}{2}, \quad j \geq j_0.$$

Using this and the maximum principle we can conclude that there exists at least one point $(\bar{x}_j, \bar{t}_j) \in \partial_p^- C_{2^l, 1}^-$ such that

$$\tilde{v}_j(\bar{x}_j, \bar{t}_j) = w_j(\bar{x}_j, \bar{t}_j) \geq \frac{1}{2}, \quad j \geq j_0.$$

Thereafter we decompose $\tilde{v}_j = \check{v}_j + \hat{v}_j$ where

$$\begin{cases} \tilde{\mathcal{H}}_j \check{v}_j = 0 & \text{in } C_{2^l, 2}^-, \\ \check{v}_j = \tilde{v}_j & \text{on } \partial_p C_{2^l, 2}^-, \end{cases} \quad \begin{cases} \tilde{\mathcal{H}}_j \hat{v}_j = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^-)} & \text{in } C_{2^l, 2}^-, \\ \hat{v}_j = 0 & \text{on } \partial_p C_{2^l, 2}^-. \end{cases}$$

As in (4.13) we see that $|\hat{v}_j(x, t)| \leq 1/4$ in $C_{2^l, 2}^-$ if j is sufficiently large and hence we can conclude that

$$\check{v}_j(\bar{x}_j, \bar{t}_j) \geq \frac{1}{4}, \quad j \geq j_0.$$

Using Lemma 3.2 we can therefore conclude that

$$\inf_{C_{1/2}^-} \check{v}_j \geq \frac{1}{4\tilde{c}}.$$

Further, since $\hat{v}_j \rightarrow 0$ uniformly on $C_{2^l, 2}^-$ as j goes to infinity, we conclude that

$$\inf_{C_{1/2}^-} \tilde{v}_j \geq \inf_{C_{1/2}^-} \check{v}_j - \|\hat{v}_j\|_{L^\infty} \geq \frac{1}{8\tilde{c}}$$

for any j , suitably large. In particular, this proves (4.12) and hence the proof of Lemma 4.1 is complete in the case $m = 0$.

Next, we consider the case when $\gamma = 1 + \alpha$ and $m = 1$. In this particular case we need to prove that (1.12) holds with $F = P_1^{(0,0)}\varphi$ and we assume that

$$(u, f, g, \varphi) \in \mathcal{P}_1(\mathcal{H}, \alpha, C^-, M_1, M_2, M_3).$$

As in the previous case, we divide the argument in three steps, and the proof will be similar to the one just carried out.

Step 1. By using that $\varphi(0,0) = 0$ and that $\varphi \in \mathcal{C}^{1,\alpha}(C^-)$, we note that

$$u(x, t) \geq \varphi(x, t) - \varphi(0, 0) \geq F(x, t) - M_3 \|(x, t)\|_p^\gamma, \quad (x, t) \in C^-.$$

As previously, we assume that (1.12) is false. That is, for every $j \in \mathbf{N}$ there exist a (smallest) positive integer k_j and $(u_j, f_j, g_j, \varphi_j) \in \mathcal{P}_1(\mathcal{H}, \alpha, C^-, M_1, M_2, M_3)$ such that $u_j(0, 0) \geq 0$ and

$$S_{k_j+1}^-(u_j - F_j) > \max \left(\frac{jM_3}{2^{(k_j+1)\gamma}}, \frac{S_{k_j}^-(u_j - F_j)}{2^\gamma}, \frac{S_{k_j-1}^-(u_j - F_j)}{2^{2\gamma}}, \dots, \frac{S_0^-(u_j - F_j)}{2^{(k_j+1)\gamma}} \right). \quad (4.16)$$

Here $F_j = P_1^{(0,0)}\varphi_j$ which in view of Theorem 2.2 is given by

$$F_j(x, t) = \sum_{i=1}^q X_i \varphi_j(0, 0) \cdot x_i.$$

Using (1.9) we see that there exists (x_j, t_j) in the closure of $C_{2^{-(k_j+1)}}^-$ such that $|u_j(x_j, t_j) - F_j(x_j, t_j)| = S_{k_j+1}^-(u_j)$ for every $j \geq 1$. Moreover, by definition, $u_j(x_j, t_j) - F_j(x_j, t_j) > 0$. Since $u_j - F_j$ is bounded and using (4.16) we conclude that $j2^{-(k_j+1)\gamma}$ is bounded, and hence, that $k_j \rightarrow \infty$ as $j \rightarrow \infty$.

Step 2. We define $(\tilde{x}_j, \tilde{t}_j) = \delta_{2^{k_j}}(x_j, t_j)$ and $\tilde{u}_j : C_{2^{k_j}}^- \rightarrow \mathbf{R}$,

$$\tilde{u}_j(x, t) = \frac{(u_j - F_j)(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^-(u_j - F_j)}.$$

As before, we note that $(\tilde{x}_j, \tilde{t}_j)$ belongs to the closure of $C_{1/2}^-$ and that

$$\tilde{u}_j(\tilde{x}_j, \tilde{t}_j) = 1.$$

Similarly we define

$$\tilde{f}_j(x, t) = 2^{-2k_j} \frac{(f_j - \mathcal{H}F_j)(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^-(u_j - F_j)}, \quad \tilde{g}_j(x, t) = \frac{(g_j - F_j)(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^-(u_j - F_j)}, \quad \tilde{\varphi}_j(x, t) = \frac{(\varphi_j - F_j)\delta_{2^{-k_j}}(x, t)}{S_{k_j+1}^-(u_j - F_j)}$$

and consider the scaled operator $\tilde{\mathcal{H}}_j = \mathcal{H}_{2^{-k_j}}$, see (2.17). We also note that \tilde{u}_j solves

$$\begin{cases} \max \left\{ \tilde{\mathcal{H}}_j \tilde{u}_j - \tilde{f}_j, \tilde{\varphi}_j - \tilde{u}_j \right\} = 0 & \text{in } C_{2^{k_j}}^-, \\ \tilde{u}_j = \tilde{g}_j & \text{on } \partial_p C_{2^{k_j}}^-. \end{cases}$$

Moreover, for any $l \in \mathbf{N}$ with $l < k_j$ we have $(\tilde{u}_j, \tilde{f}_j, \tilde{u}_j, \tilde{\varphi}_j) \in \mathcal{P}_1(\tilde{\mathcal{H}}_j, \alpha, C_{2^l}^-, \tilde{M}_1^j, \tilde{M}_2^j, \tilde{M}_3^j)$. This time we prove our claim by using the seminorm

$$N_3^j := \sum_{i=1}^q \left| X_i^{2^{-k_j}} \tilde{\varphi}_j(0, 0) \right| + \sup_{z, \zeta \in C_{2^l}^-, z \neq \zeta} \frac{\left| \tilde{\varphi}_j(z) - \tilde{\varphi}_j(\zeta) - \sum_{i=1}^q (z_i - \zeta_i) X_i^{2^{-k_j}} \tilde{\varphi}_j(\zeta) \right|}{d_p(z, \zeta)^\gamma}, \quad (4.17)$$

where $X_i^{2^{-k_j}}$, $i = 1, 2, \dots, q$, are the scaled vector fields defined as in Subsection 2.3. By definition,

$$N_3^j + \|\tilde{\varphi}_j\|_{\mathcal{C}^{0,\alpha}(C_{2^l}^-)} \leq \tilde{M}_3^j,$$

where the Hölder norm is defined in terms of the scaled vector fields $\{X_i^{2^{-k_j}}\}$. A direct calculation shows that

$$\mathcal{H}F_j(x, t) = \sum_{i=1}^q b_i(x, t) X_i \left(\sum_{j=1}^q X_j \varphi(0, 0) \cdot x_j \right) = \sum_{i,j=1}^q b_i(x, t) X_j \varphi(0, 0) X_j x_i = \sum_{i,j=1}^q c_{ji}(x) b_i(x, t) X_j \varphi(0, 0),$$

and therefore

$$\tilde{M}_1^j \leq 2^{(l+1)\gamma}, \quad \tilde{M}_2^j \leq 2^{-2k_j} \frac{M_2 + c(\mathbf{c}_\alpha, X)M_3}{S_{k_j+1}^-(u_j - F_j)} \quad \text{and} \quad N_3^j \leq 2^{(l-k_j)\gamma} \frac{M_3}{S_{k_j+1}^-(u_j - F_j)}.$$

Note that the first two inequalities follow directly from the definition while the third one follows from the fact that $X_i \tilde{\varphi}_j(0, 0) = 0$ and the following calculation:

$$\begin{aligned} & \left| (\varphi_j - F_j)(z) - (\varphi_j - F_j)(\zeta) - \sum_{i=1}^q (x_i - \xi_i) X_i (\varphi_j - F_j)(\zeta) \right| \\ & \leq \|\varphi_j\|_{C^{1,\alpha}(C_{2^l}^-)} d_p(z, \zeta)^{1+\alpha} + \left| F_j(z) - F_j(\zeta) - \sum_{i=1}^q (x_i - \xi_i) X_i F_j(\zeta) \right| \\ & = \|\varphi_j\|_{C^{1,\alpha}(C_{2^l}^-)} d_p(z, \zeta)^{1+\alpha}. \end{aligned}$$

We also mention that the last equality follows from the definition of F_j and the fact that $X_i x_j = \delta_{ij}$ when X_i belongs to the first layer of the Jacobian basis of \mathbf{G} . Therefore, by (4.16),

$$\lim_{j \rightarrow \infty} \tilde{M}_2^j = \lim_{j \rightarrow \infty} N_3^j = 0.$$

Step 3. First of all, $\|\tilde{\varphi}_j\|_{L^\infty} \leq N_3^j$, and this follows from the definition of N_3^j in (4.17) with $\zeta = 0$. From here this step is completely analogous to Step 3 in the first part of the proof, that is when $m = 0$ and $\gamma = \alpha$, but with \tilde{M}_3^j replaced by N_3^j . This completes the proof when $m = 1$, $\gamma = 1 + \alpha$. It remains to consider the case when $m = 2$ and $\gamma = 2$. In this particular case we need to prove that (1.12) holds with $F = \varphi$ and we assume that

$$(u, f, g, \varphi) \in \mathcal{P}_2(\mathcal{H}, \alpha, C^-, M_1, M_2, M_3).$$

Below we will assume, without loss of generality, that $\varphi \equiv 0$. Indeed, since $\varphi \in C^{2,\alpha}(C^-)$, $\mathcal{H}\varphi \in C^{0,\alpha}(C^-)$ and we can consider the modified functions $\hat{u} = u - \varphi$, $\hat{f} = f - \mathcal{H}\varphi$, $\hat{g} = g - \varphi$ in C^- . Note that

$$(\hat{u}, \hat{f}, \hat{g}, 0) \in \mathcal{P}_2(\mathcal{H}, \alpha, C^-, \widehat{M}_1, \widehat{M}_2, 0),$$

where \widehat{M}_1 and \widehat{M}_2 (only) depend on M_1 , M_2 and M_3 . In this setting, we can now argue as before, but this time, the contradictive assumption is that for every $j \in \mathbf{N}$ there exist a positive integer k_j and functions $(u_j, f_j, g_j, 0) \in \mathcal{P}_2(\mathcal{H}, \alpha, C^-, M_1, M_2, 0)$ such that

$$S_{k_j+1}^-(u_j) > \max \left(\frac{j}{2^{(k_j+1)\gamma}}, \frac{S_{k_j}^-(u_j)}{2^\gamma}, \frac{S_{k_j-1}^-(u_j)}{2^{2\gamma}}, \dots, \frac{S_0^-(u_j)}{2^{(k_j+1)\gamma}} \right). \quad (4.18)$$

Using (1.9) we see that there exists (x_j, t_j) in the closure of $C_{2^{-(k_j+1)}}^-$ such that $|u_j(x_j, t_j)| = S_{k_j+1}^-(u_j)$ for every $j \geq 1$. Moreover, by definition, $u_j(x_j, t_j) > 0$. Using (4.18) we can conclude, as u_j is bounded, that $j2^{-(k_j+1)\gamma}$ is bounded and hence that $k_j \rightarrow \infty$ as $j \rightarrow \infty$. From now on, the proof follows along the same lines as in the case $m = 0$, $\gamma = \alpha$ and we omit the details. This completes the proof. \square

4.2 Proof of Lemma 4.2

Assume $(u, f, g, \varphi) \in \tilde{\mathcal{P}}_m(\mathcal{H}, \alpha, C^+, M_1, M_2, M_3, M_4)$. Then we first note that there exists a constant $\eta_{m,\alpha} = \eta(\mathcal{H}, m, \alpha, M_1, M_2, M_3)$ such that

$$\inf_{C_r^+} (u - g) \geq -\eta_{m,\alpha} r^\gamma, \quad r \in (0, 1), \quad (4.19)$$

where $\gamma = m + \alpha$ for $m = 0, 1$ and $\gamma = 2$ for $m = 2$. In fact, to prove this we let v be the solution to the Dirichlet problem (2.15) in the domain $D = C^+$ with right-hand side defined by f and with boundary defined by g . Then

$$(v, f, g) \in \mathcal{D}_m(\mathcal{H}, \alpha, C^+, M_1, M_2, M_3).$$

Furthermore, since v solves the Dirichlet problem while u solves the obstacle problem with the same boundary data we have $u \geq v$ on the closure of C^+ and hence (4.19) follows directly if we apply Lemma 3.10 to v . We next note, as in the proof of Lemma 3.10, that it suffices to prove that

$$\sup_{C_r^+} |u - P_m^{(0,0)} g| \leq cr^\gamma, \quad r \in (0, 1), \quad (4.20)$$

where $\gamma = m + \alpha$ for $m = 0, 1$ and

$$\sup_{C_r^+} |u - g| \leq cr^\gamma, \quad r \in (0, 1), \quad (4.21)$$

where $\gamma = 2$ for $m = 2$. Furthermore, as in the proof of Lemma 3.10, we can without loss of generality assume that $P_m^{(0,0)} g \equiv 0$ for $m = 0, 1$ and that $g \equiv 0$ for $m = 2$. To prove (4.20) and (4.21) after these preliminary problem reduction steps we define $S_k^+(u)$ as in (1.10) and we intend to prove that there exists a positive constant $\tilde{c} = \tilde{c}(\mathcal{H}, m, \alpha, M_1, M_2, M_3, M_4)$ such that (1.14) holds for all $k \in \mathbf{N}$ with F as defined in (1.13) (which by our problem reduction reads $F \equiv 0$). Again, if (1.14) holds then

$$S_k^+(u - F) \leq \frac{\tilde{c}}{2^{k\gamma}}$$

and hence (4.20) and (4.21), and Lemma 4.2, follow.

We first consider the case $m = 0$ and prove (1.14) with $\gamma = \alpha$. In particular, we assume that

$$(u, f, g, \varphi) \in \tilde{\mathcal{P}}_0(\mathcal{H}, \alpha, C^+, M_1, M_2, M_3, M_4),$$

and, as in the proof of Lemma 4.1 and following the lines of [NPP10], we divide the argument into three steps.

Step 1. We first note, using (4.19), that

$$u(x, t) \geq -(\eta_{m,\alpha} + M_3) \|(x, t)\|_p^\gamma, \quad (x, t) \in C^+. \quad (4.22)$$

Assume that (1.14) is false. Then for every $j \in \mathbf{N}$, there exists a positive integer k_j and functions $(u_j, f_j, g_j, \varphi_j) \in \tilde{\mathcal{P}}_0(\mathcal{H}, \alpha, C^+, M_1, M_2, M_3, M_4)$ such that $u_j(0, 0) = 0 \geq \varphi_j(0, 0)$ and

$$S_{k_j+1}^+(u_j) > \max \left(\frac{j(\eta_{m,\alpha} + M_3)}{2^{(k_j+1)\gamma}}, \frac{S_{k_j}^+(u_j)}{2^\gamma}, \frac{S_{k_j-1}^+(u_j)}{2^{2\gamma}}, \dots, \frac{S_0^+(u_j)}{2^{(k_j+1)\gamma}} \right). \quad (4.23)$$

Using the definition of $S_{k_j+1}^+$ in (1.10) we see that there exists (x_j, t_j) in the closure of $C_{2^{-(k_j+1)}}^+$ such that $|u_j(x_j, t_j)| = S_{k_j+1}^+(u_j)$ for every $j \geq 1$. Moreover from (4.22) and (4.23) it follows that $u_j(x_j, t_j) > 0$. Using (4.23) we can conclude, as $|u_j| \leq M_1$, that $j2^{-\gamma k_j}$ is bounded and hence that $k_j \rightarrow \infty$ as $j \rightarrow \infty$.

Step 2. We define $(\tilde{x}_j, \tilde{t}_j) = \delta_{2^{k_j}}((x_j, t_j))$ and $\tilde{u}_j : C_{2^{k_j}}^+ \rightarrow \mathbf{R}$,

$$\tilde{u}_j(x, t) = \frac{u_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^+(u_j)}.$$

Note that $(\tilde{x}_j, \tilde{t}_j)$ belongs to the closure of $C_{1/2}^+$ and

$$\tilde{u}_j(\tilde{x}_j, \tilde{t}_j) = 1. \quad (4.24)$$

Moreover, we let $\tilde{\mathcal{H}}_j = \mathcal{H}_{2^{-k_j}}$, see (2.17) for the exact definition of this scaled operator, and

$$\tilde{f}_j(x, t) = 2^{-2k_j} \frac{f_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^+(u_j)}, \quad \tilde{g}_j(x, t) = \frac{g_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^+(u_j)}, \quad \tilde{\varphi}_j(x, t) = \frac{\varphi_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^+(u_j)} \quad (4.25)$$

whenever $(x, t) \in C_{2^{k_j}}^+$. Then, using (2.16), we see that

$$\begin{cases} \max\{\tilde{\mathcal{H}}_j \tilde{u}_j - \tilde{f}_j, \tilde{\varphi}_j - \tilde{u}_j\} = 0, & \text{in } C_{2^{k_j}}^+, \\ \tilde{u}_j = \tilde{g}_j, & \text{on } \partial_p C_{2^{k_j}}^+. \end{cases}$$

In the following we let $l \in \mathbf{N}$ be fixed and to be specified below. From (4.23) it follows that

$$\sup_{C_{2^l}^+} |\tilde{u}_j| = \frac{S_{k_j-l}^+(u_j)}{S_{k_j+1}^+(u_j)} \leq 2^{(l+1)\gamma} \quad \text{whenever } k_j > l, \quad (4.26)$$

and that

$$(\tilde{u}_j, \tilde{f}_j, \tilde{u}_j, \tilde{\varphi}_j) \in \tilde{\mathcal{P}}_0(\tilde{\mathcal{H}}_j, \alpha, C_{2^l}^+, \tilde{M}_1^j, \tilde{M}_2^j, \tilde{M}_3^j, \tilde{M}_4^j),$$

for some $\tilde{M}_1^j, \tilde{M}_2^j, \tilde{M}_3^j, \tilde{M}_4^j$. Furthermore, using (4.25), (4.26) and the continuity of \tilde{f}_j , we have

$$\tilde{M}_1^j \leq 2^{(l+1)\gamma}, \quad \tilde{M}_2^j \leq 2^{-2k_j} \frac{M_2}{S_{k_j+1}^+(u_j)}. \quad (4.27)$$

Moreover, we let

$$m_j = \max \left\{ \|\tilde{g}_j\|_{L^\infty(C_{2^l}^+)}, \sup_{C_{2^l}^+} \tilde{\varphi}_j \right\}.$$

Then, using (4.23) and the $C_X^{0,\alpha}$ -regularity of g_j and φ_j , we see that

$$\lim_{j \rightarrow \infty} \tilde{M}_2^j = \lim_{j \rightarrow \infty} m_j = 0. \quad (4.28)$$

Note that we can not ensure the decay of \tilde{M}_4^j , as $j \rightarrow \infty$, as we only know that $\tilde{\varphi}_j(0, 0) \leq 0$.

Step 3. Below we will choose l suitably large to find a contradiction and we will use the following short notation for $R, R_1, R_2 \geq 1$,

$$\begin{aligned} \partial_p^+ C_R^+ &= \partial_p C_R^+ \cap \{t > 0\}, \quad \partial_p^- C_R^+ = \partial_p C_R^+ \cap \{t = 0\}, \\ \partial_p^+ C_{R_1, R_2}^+ &= \partial_p C_{R_1, R_2}^+ \cap \{t > 0\}, \quad \partial_p^- C_{R_1, R_2}^+ = \partial_p C_{R_1, R_2}^+ \cap \{t = 0\}. \end{aligned} \quad (4.29)$$

Above C_{R_1, R_2}^+ denotes $B_d(0, R_1) \times (0, R_2]$. We now consider $j_0 \in \mathbf{N}$ such that $k_j > 2^l$ for $j \geq j_0$ and we let \tilde{v}_j be the solution to

$$\begin{cases} \tilde{\mathcal{H}}_j \tilde{v}_j = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^+)} & \text{in } C_{2^l}^+, \\ \tilde{v}_j = \tilde{M}_1^j & \text{on } \partial_p^+ C_{2^l}^+, \\ \tilde{v}_j = m_j & \text{on } \partial_p^- C_{2^l}^+. \end{cases}$$

To continue we prove that

$$\tilde{u}_j \leq \tilde{v}_j \quad \text{in } C_{2^l}^+, \quad (4.30)$$

and that this contradicts (4.24). Indeed, by the maximum principle we have $\tilde{v}_j \geq m_j \geq \tilde{\varphi}_j$ in $C_{2^l}^+$. Furthermore

$$\tilde{\mathcal{H}}_j(\tilde{v}_j - \tilde{u}_j) = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^+)} + \tilde{f}_j \leq 0 \quad \text{in } D := C_{2^l}^+ \cap \{(x, t) : \tilde{u}_j(x, t) > \tilde{\varphi}_j(x, t)\},$$

and $\tilde{v}_j \geq \tilde{u}_j$ on $\partial_p D$. Hence (4.30) follows from the maximum principle. Next, we show that (4.30) contradicts (4.24). We write $\tilde{v}_j = w_j + \tilde{w}_j + \hat{w}_j$ on $C_{2^l, 1}^+$ where

$$\begin{cases} \tilde{\mathcal{H}}_j w_j = 0 & \text{in } C_{2^l, 1}^+, \\ w_j = 0 & \text{on } \partial_p^+ C_{2^l, 1}^+, \\ w_j = \tilde{v}_j & \text{on } \partial_p^- C_{2^l, 1}^+, \end{cases} \quad \begin{cases} \tilde{\mathcal{H}}_j \tilde{w}_j = 0 & \text{in } C_{2^l, 1}^+, \\ \tilde{w}_j = \tilde{v}_j & \text{on } \partial_p^+ C_{2^l, 1}^+, \\ \tilde{w}_j = 0 & \text{on } \partial_p^- C_{2^l, 1}^+, \end{cases} \quad \begin{cases} \tilde{\mathcal{H}}_j \hat{w}_j = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^+)} & \text{in } C_{2^l, 1}^+, \\ \hat{w}_j = 0 & \text{on } \partial_p C_{2^l, 1}^+. \end{cases}$$

By the maximum principle we first see that

$$w_j \leq m_j \quad \text{in } C_{2^l, 1}^+, \quad (4.31)$$

and that

$$\|\hat{w}_j\|_{L^\infty(C_{2^l,1}^+)} \leq \|\tilde{f}_j\|_{L^\infty(C_{2^l,1}^+)} \leq \tilde{M}_2^j. \quad (4.32)$$

We next use Lemma 3.9 in the cylinder $C_{2^l,1}^+$, and by (4.27) we see that

$$\sup_{C^+} \tilde{w}_j \leq c2^{(Q+1)l} e^{-c4^l} \sup_{\partial_p^+ C_{2^l,1}^+} \tilde{v}_j \leq c2^{(Q+1)l} e^{-c4^l} \tilde{M}_1^j \leq c2^{(Q+1)l} e^{-c4^l} 2^{(l+1)\gamma}, \quad (4.33)$$

and, in particular, we note that the right hand side in this inequality can be made arbitrarily small by choosing l large enough, independent of j . Combining (4.31), (4.32) and (4.33) we conclude that, for a suitably large l and j_0 , we have

$$\sup_{C^+} \tilde{v}_j \leq \frac{1}{2} \quad \text{for any } j \geq j_0,$$

which contradicts (4.24) and (4.30). This completes the proof in the case $m = 0$.

It remains to consider the cases $\gamma = m + \alpha$ for $m = 1$ and $\gamma = 2$ for $m = 2$, which can be proved in complete analogy with step 1-3 above only with a slight change of motivation in (4.28). We omit the details. \square

4.3 Proof of Lemma 4.3

Using Lemma 4.1 we only need to prove the statement on C_r^+ . We also note, as in the proof of Lemma 3.10, that using the triangle inequality and Lemma 2.3 it suffices to prove that

$$\sup_{C_r^+(0,0)} \left| u - P_m^{(0,0)} \varphi \right| \leq cr^\gamma, \quad r \in (0,1), \quad (4.34)$$

for $\gamma = m + \alpha$ for $m = 0, 1$ and

$$\sup_{C_r^+(0,0)} |u - \varphi| \leq cr^\gamma, \quad r \in (0,1), \quad (4.35)$$

for $\gamma = 2$ for $m = 2$. Furthermore, as in the proof of Lemma 3.10, we can without loss of generality assume that $P_m^{(0,0)} \varphi \equiv 0$ for $m = 0, 1$ and that $\varphi \equiv 0$ for $m = 2$. To prove (4.34) and (4.35) after these preliminary problem reduction steps we define $S_k^+(u)$ as in (1.10) and we intend to prove that there exists a positive constant $\tilde{c} = \tilde{c}(\mathcal{H}, m, \alpha, M_1, M_2, M_3, M_4)$ such that (1.14) holds for all $k \in \mathbf{N}$ with F as defined in (1.13), which again, by our problem reduction reads $F \equiv 0$. If (1.14) holds then

$$S_k^+(u - F) \leq \frac{\tilde{c}}{2^{k\gamma}} \quad (4.36)$$

and hence (4.34), (4.35) and Lemma 4.3 follows.

We first consider the case $m = 0$ and prove (1.14) with $\gamma = \alpha$. In particular, we assume that

$$(u, f, g, \varphi) \in \mathcal{P}_0(\mathcal{H}, \alpha, C^+, M_1, M_2, M_3),$$

and we will proceed in a way similar to the proof Lemma 4.2. That is, we assume that (4.36) is false. Then for every $j \in \mathbf{N}$ there exist a positive integer k_j and $(u_j, f_j, g_j, \varphi_j) \in \mathcal{P}_0(\mathcal{H}, \alpha, C^+, M_1, M_2, M_3)$ such that $u_j(0,0) = \varphi_j(0,0) = 0$ and

$$S_{k_j+1}^+(u_j) > \max \left(\frac{jM_3}{2^{(k_j+1)\gamma}}, \frac{S_{k_j}^+(u_j)}{2^\gamma}, \frac{S_{k_j-1}^+(u_j)}{2^{2\gamma}}, \dots, \frac{S_0^+(u_j)}{2^{(k_j+1)\gamma}} \right).$$

Furthermore, there exists (x_j, t_j) in the closure of $C_{2^{-(k_j+1)}}^+$ such that $|u_j(x_j, t_j)| = S_{k_j+1}^+(u_j)$ for every $j \geq 1$. As in Step 2 in the proof of Lemma 4.2, we define $(\tilde{x}_j, \tilde{t}_j) = \delta_{2^{k_j}}((x_j, t_j))$ and $\tilde{u}_j : C_{2^{k_j}}^+ \rightarrow \mathbf{R}$,

$$\tilde{u}_j(x, t) = \frac{u_j(\delta_{2^{-k_j}}(x, t))}{S_{k_j+1}^+(u_j)}.$$

Note that $(\tilde{x}_j, \tilde{t}_j)$ belongs to the closure of $C_{1/2}^+$ and

$$\tilde{u}_j(\tilde{x}_j, \tilde{t}_j) = 1.$$

Also in this case we can define functions $\tilde{f}_j, \tilde{\varphi}_j$ similarly to \tilde{u}_j , see (4.25), and then

$$(\tilde{u}_j, \tilde{f}_j, \tilde{u}_j, \tilde{\varphi}_j) \in \mathcal{P}_0(\tilde{\mathcal{H}}_j, \alpha, C_{2^l}^+, \tilde{M}_1^j, \tilde{M}_2^j, \tilde{M}_3^j),$$

where

$$\tilde{M}_1^j \leq 2^{(l+1)\gamma}, \quad \tilde{M}_2^j \leq 2^{-2k_j} \frac{M_2}{S_{k_j+1}^-(u_j)}, \quad \tilde{M}_3^j \leq 2^{(l-k_j)\gamma} \frac{M_3}{S_{k_j+1}^-(u_j)}.$$

We now intend to complete the argument by a contradiction. We fix a suitable positive integer l , to be specified below, and we consider $j_0 \in \mathbf{N}$ such that $k_j > 2^l$ for $j > j_0$. Then as in Lemma 4.1 we prove that

$$v_j \leq \tilde{u}_j \leq \tilde{v}_j, \quad \text{in } C_{2^l}^+, \quad (4.37)$$

where v_j and \tilde{v}_j are defined by

$$\begin{cases} \tilde{\mathcal{H}}_j v_j = \|\tilde{f}_j\|_{L^\infty(C_{2^l}^+)} & \text{in } C_{2^l}^+, \\ v_j = \hat{g}_j & \text{on } \partial_p C_{2^l}^+, \end{cases} \quad \text{respectively} \quad \begin{cases} \tilde{\mathcal{H}}_j \tilde{v}_j = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^+)} & \text{in } C_{2^l}^+, \\ \tilde{v}_j = \max\{\hat{g}_j, \tilde{M}_3^j\} & \text{on } \partial_p C_{2^l}^+, \end{cases} \quad (4.38)$$

for some function \hat{g}_j which coincides with \tilde{u}_j on $\partial_p C_{2^l}^+$. Moreover, we note that

$$\lim_{j \rightarrow \infty} \tilde{M}_3^j = 0 = \lim_{j \rightarrow \infty} \|\hat{g}_j\|_{L^\infty(\partial_p C_{2^l}^+ \cap \{t=0\})} = 0. \quad (4.39)$$

We next show that (4.37)-(4.39) leads to a contradiction. To do this we recall the notation introduced in (4.29) and write $\tilde{v}_j = w_j + \tilde{w}_j + \hat{w}_j$ on $C_{2^l,1}^+$ where

$$\begin{cases} \tilde{\mathcal{H}}_j w_j = 0 & \text{in } C_{2^l,1}^+, \\ w_j = 0 & \text{on } \partial_p^+ C_{2^l,1}^+, \\ w_j = \tilde{v}_j & \text{on } \partial_p^- C_{2^l,1}^+, \end{cases} \quad \begin{cases} \tilde{\mathcal{H}}_j \tilde{w}_j = 0 & \text{in } C_{2^l,1}^+, \\ \tilde{w}_j = \tilde{v}_j & \text{on } \partial_p^+ C_{2^l,1}^+, \\ \tilde{w}_j = 0 & \text{on } \partial_p^- C_{2^l,1}^+, \end{cases} \quad \begin{cases} \tilde{\mathcal{H}}_j \hat{w}_j = -\|\tilde{f}_j\|_{L^\infty(C_{2^l}^+)} & \text{in } C_{2^l,1}^+, \\ \hat{w}_j = 0 & \text{on } \partial_p C_{2^l,1}^+. \end{cases}$$

Arguing as above (4.15),

$$\|\hat{w}_j\|_{L^\infty(C_{1/2}^+)} \leq \tilde{M}_2^j, \quad \|\tilde{w}_j\|_{L^\infty(C_{1/2}^+)} \leq c 2^{(Q+1)l} e^{-c 4^l} \left(\max\{2^{(l+1)\gamma}, \tilde{M}_3^j\} + 2^{2l} \tilde{M}_2^j \right).$$

Since $\tilde{v}_j(\tilde{x}_j, \tilde{t}_j) \geq \tilde{u}_j(\tilde{x}_j, \tilde{t}_j) = 1$ we can conclude, by choosing l large enough, that both $\|\hat{w}_j\|_{L^\infty(C_{1/2}^+)}$ and $\|\tilde{w}_j\|_{L^\infty(C_{1/2}^+)}$ tend to zero as $j \rightarrow \infty$, so that

$$w_j(\tilde{x}_j, \tilde{t}_j) \geq \frac{1}{2} \quad \text{for } j \geq j_0.$$

Now, applying the maximum principle we can conclude that there exists at least one point $(\bar{x}, \bar{t}) \in \partial_p^- C_{2^l,1}^+$ such that

$$\tilde{v}_j(\bar{x}, \bar{t}) = w_j(\bar{x}, \bar{t}) \geq \frac{1}{2} \quad \text{for } j \geq j_0,$$

which contradicts (4.38) and (4.39). Hence (4.34) holds for $m = 0$, $\gamma = \alpha$.

To prove Lemma 4.3 when $m = 1$, $\gamma = 1 + \alpha$ or when $m = 2$, $\gamma = 2$ we can use the same arguments as above and we omit the details. \square

5 Proof of Theorem 1.1

The proof might be somewhat difficult to get an overview of although straightforward, since we have to divide it into different cases along the way. Therefore we will deal with different choices of m , i.e., $m = 0, 1, 2$, in the subsections below.

5.1 Proof of Theorem 1.1 when $m = 0$, $\gamma = \alpha$

We will divide the proof into different cases stemming from geometric assumptions. To explain this further we take $R \in (0, 1)$ and a constant $C_1 > 1$ such that $C_{2R} \subseteq C_{2C_1R}(\hat{x}, \hat{t}) \subseteq C$ for all $(\hat{x}, \hat{t}) \in C_{2R}$ and we define $\mathcal{F} = \overline{C}_{2R} \cap \{(x, t) : u(x, t) = \varphi(x, t)\}$. Our goal is to prove that there exists a constant $\hat{c} = \hat{c}(\mathcal{H}, \alpha, f, \varphi)$ such that

$$\sup_{\substack{(x,t), (\hat{x}, \hat{t}) \in C_R \\ (x,t) \neq (\hat{x}, \hat{t})}} \frac{|u(x, t) - u(\hat{x}, \hat{t})|}{d_p((x, t), (\hat{x}, \hat{t}))^\alpha} \leq \hat{c}. \quad (5.1)$$

Before we continue, we note that if $\mathcal{F} = \emptyset$ then u is a solution to the Dirichlet problem and we can apply Lemma 3.3 and the result follows immediately. On the other hand, if $(\hat{x}, \hat{t}) \in C_{2R} \cap \mathcal{F}$, then we can translate both the operator and the functions so that (\hat{x}, \hat{t}) is our new origin and after this reformulation we can use Lemma 4.3 to obtain (5.1), but this time for $(x, t) \in C_{2C_1R}(\hat{x}, \hat{t})$. Obviously the analogue of this estimate holds whenever $(x, t) \in C_{2R} \cap \mathcal{F}$. In view of this it remains to consider the case when $(x, t), (\hat{x}, \hat{t}) \in C_R \setminus \mathcal{F}$. When we continue we divide the proof in two cases depending on how far apart the points $(x, t), (\hat{x}, \hat{t})$ are compared with the distance from (x, t) to \mathcal{F} . For this reason we define $r = d_p((x, t), \mathcal{F}) := \inf\{d_p((x, t), (\xi, \tau)) : (\xi, \tau) \in \mathcal{F}\}$ and for fixed (x, t) we let $(\tilde{x}, \tilde{t}) \in \mathcal{F}$ be such a minimizing point.

Case 1: Assume that $(\hat{x}, \hat{t}) \in C_R \setminus C_{r/2}(x, t)$, then $d_p((x, t), (\hat{x}, \hat{t})) > c_0 r$ for some constant c_0 . Using the triangle inequality we get

$$d_p((\hat{x}, \hat{t}), (\tilde{x}, \tilde{t})) \leq \left(1 + \frac{1}{c_0}\right) d_p((x, t), (\hat{x}, \hat{t})) \quad (5.2)$$

and by using (5.1) we get

$$\begin{aligned} |u(x, t) - u(\hat{x}, \hat{t})| &\leq |u(x, t) - u(\tilde{x}, \tilde{t})| + |u(\tilde{x}, \tilde{t}) - u(\hat{x}, \hat{t})| \\ &\leq \hat{c} (d_p((x, t), (\hat{x}, \hat{t}))^\alpha + d_p((\hat{x}, \hat{t}), (\tilde{x}, \tilde{t}))^\alpha) \\ &\leq c_1 d_p((x, t), (\hat{x}, \hat{t}))^\alpha. \end{aligned}$$

As we are about to prove (5.1) we remark that we may use (5.1) in the computation above since $(\tilde{x}, \tilde{t}) \in C_{2R} \cap \mathcal{F}$, and in that particular case we have proved that (5.1) is valid. Hence, (5.1) holds also in this case.

Case 2: Assume that $(\hat{x}, \hat{t}) \in C_{r/2}(x, t)$. Then we define a new function

$$v(\hat{x}, \hat{t}) := u(\hat{x}, \hat{t}) - u(\tilde{x}, \tilde{t}),$$

and we remark that $u(\tilde{x}, \tilde{t})$ is to be treated as a constant. Then, since $(\tilde{x}, \tilde{t}) \in C_{2R} \cap \mathcal{F}$ we can use (5.1) once more and we see that

$$|v(\hat{x}, \hat{t})| \leq c d_p((\hat{x}, \hat{t}), (\tilde{x}, \tilde{t}))^\alpha \leq c_2 r^\alpha.$$

To proceed, we define the function

$$w(y, s) := \frac{v^{r, (x, t)}(y, s)}{r^\alpha}$$

where $v^{r, (x, t)}$ is defined as in (2.18). For $(y, s) \in C_{1/2}$ the following holds:

- i) $|w(y, s)| \leq c_3$ for some constant c_3 ,
- ii) $\mathcal{H}^{r, (x, t)} w(y, s) = r^{2-\alpha} f^{r, (x, t)}(y, s)$.

The Schauder estimate in Lemma 3.3 then implies that

$$|w(y, s) - w(0, 0)| \leq c_4 d_p((0, 0), (y, s))^\alpha$$

for some positive constant c_4 . By the definition of w this concludes the proof in the case $m = 0$.

5.2 Proof of Theorem 1.1 when $m = 1$, $\gamma = 1 + \alpha$

As before we take $R \in (0, 1)$ and $C_1 > 1$ such that $C_{2R} \subseteq C_{2C_1 R}(\hat{x}, \hat{t}) \subseteq C$ for all $(\hat{x}, \hat{t}) \in C_{2R}$ and we define $\mathcal{F} = \overline{C_{2R}} \cap \{(x, t) : u(x, t) = \varphi(x, t)\}$. We assume that $\mathcal{F} \neq \emptyset$ since in that case we can use Lemma 3.3 directly to obtain the desired result. We have already shown that $u \in \mathcal{C}_X^{0, \alpha}(C_R)$ so we are left with the task to prove that

$$\|X_i u\|_{\mathcal{C}_X^{0, \alpha}(C_R)} = \sup_{C_R} |X_i u| + \sup_{C_R} \frac{|X_i u(x, t) - X_i u(\hat{x}, \hat{t})|}{d_p((x, t), (\hat{x}, \hat{t}))^\alpha} \leq \hat{c} \quad (5.3)$$

and that

$$\sup_{C_R} \left| X_i u(x, t) - X_i u(\tilde{x}, \tilde{t}) - \sum_{i=1}^q X_i u(\tilde{x}, \tilde{t})(x_i - \tilde{x}_i) \right| \leq \hat{c} d_p((x, t), (\tilde{x}, \tilde{t}))^{1+\alpha} \quad (5.4)$$

for $i \in \{1, 2, \dots, q\}$ and for some constant $\hat{c} = \hat{c}(\mathcal{H}, \alpha, f, \varphi)$.

Step 1. Notice that, as a consequence of Lemma 4.3 and the fact that $u \geq \varphi$, we know that $X_i u(x, t) = X_i \varphi(x, t)$ whenever $(x, t) \in \mathcal{F}$. Now, let $(\hat{x}, \hat{t}) \in C_{2R} \cap \mathcal{F}$ and define the function

$$v(x, t) := u(x, t) - u(\hat{x}, \hat{t}) - \sum_{i=1}^q X_i u(\hat{x}, \hat{t})(x_i - \hat{x}_i). \quad (5.5)$$

We note that, in fact $v(x, t) = u(x, t) - P_1^{(\hat{x}, \hat{t})} \varphi(x, t)$ and we may use Lemma 4.3 to deduce that

$$|v(x, t)| \leq c_1 d_p((x, t), (\hat{x}, \hat{t}))^{1+\alpha},$$

for all $(x, t) \in C_R$. That is, (5.4) holds when the supremum is taken over the set where $(x, t) \in C_R \setminus \mathcal{F}$, $(\hat{x}, \hat{t}) \in C_R \cap \mathcal{F}$.

Step 2. Now we intend to prove that (5.3) also holds for $(x, t) \in C_R \setminus \mathcal{F}$, $(\hat{x}, \hat{t}) \in C_R \cap \mathcal{F}$. To do this let $r := d_p((x, t), \mathcal{F})$ and consider the case when $d_p((x, t), (\hat{x}, \hat{t})) \leq 2r$. Then, for v defined as in (5.5) we define the function

$$w(y, s) := \frac{v^{r, (x, t)}(y, s)}{r^{1+\alpha}} \quad \text{on } C_1. \quad (5.6)$$

This function satisfies the following on $C_{1/2}$, using the notation introduced in Section 2.3,

- i) $|w| \leq c_2$ for some constant c_2 , independent of r ,
- ii) $\mathcal{H}^{r, (x, t)} w(y, s) = r^{1-\alpha} \left(f^{r, (x, t)} - \sum_{i=1}^q b_i^{r, (x, t)} X_i \varphi(\hat{x}, \hat{t}) \right).$

By using the Schauder estimate in Lemma 3.3 we see that

$$\|X_i w\|_{\mathcal{C}_X^{0, \alpha}(C_R)} \leq c'_2, \quad i \in \{1, 2, \dots, q\}, \quad (5.7)$$

and since

$$X_i w(0, 0) = \frac{X_i u(x, t) - X_i u(\hat{x}, \hat{t})}{r^{1+\alpha}}, \quad (5.8)$$

this implies the Hölder continuity of $X_i u$. Furthermore, we may combine (5.7), (5.8) and the fact that $X_i u(\hat{x}, \hat{t}) = X_i \varphi(\hat{x}, \hat{t})$ with $r = R$ to obtain

$$\sup_{C_R} |X_i u| \leq c'_2 + \sup_{C_R} |\varphi|.$$

On the other hand, if $d_p((x, t), (\hat{x}, \hat{t})) > 2r$ and if $(\tilde{x}, \tilde{t}) \in \mathcal{F}$ is such that $r = d_p((x, t), (\tilde{x}, \tilde{t}))$ we use the triangle inequality to obtain

$$\begin{aligned} d_p((\hat{x}, \hat{t}), (\tilde{x}, \tilde{t})) &\leq d_p((x, t), (\hat{x}, \hat{t})) + d_p((x, t), (\tilde{x}, \tilde{t})) \\ &\leq 3/2 d_p((x, t), (\hat{x}, \hat{t})). \end{aligned}$$

Hence,

$$\begin{aligned}
|X_i u(x, t) - X_i u(\hat{x}, \hat{t})| &\leq |X_i u(x, t) - X_i u(\tilde{x}, \tilde{t})| + |X_i u(\tilde{x}, \tilde{t}) - X_i u(\hat{x}, \hat{t})| \\
&\leq d_p((x, t), (\tilde{x}, \tilde{t}))^\alpha + |X_i \varphi(\tilde{x}, \tilde{t}) - X_i \varphi(\hat{x}, \hat{t})| \\
&\leq c'_3 d_p((x, t), (\hat{x}, \hat{t}))^\alpha,
\end{aligned}$$

where we also used the definition of r and the fact that $X_i u(x, t) = X_i \varphi(x, t)$ whenever $(x, t) \in \mathcal{F}$. That is, we have proved that (5.3) holds when the supremum is taken over the set where $(x, t) \in C_R \setminus \mathcal{F}$, $(\hat{x}, \hat{t}) \in \mathcal{F} \cap C_R$.

Step 3. In this step, which completes the proof of (5.3), we will prove that the statement also is valid for $(x, t), (\hat{x}, \hat{t}) \in C_R \setminus \mathcal{F}$. As before, we let r and $(\tilde{x}, \tilde{t}) \in C_{2R} \cap \mathcal{F}$ be such that $r = d_p((x, t), \mathcal{F}) = d_p((x, t), (\tilde{x}, \tilde{t}))$ and firstly we consider the case when $(\hat{x}, \hat{t}) \in C_r(x, t)$. In analogy with (5.6) we define a function w but we replace (\hat{x}, \hat{t}) with (x, t) in (5.5). Using the same argument as above we find that (5.7) still holds and in particular

$$|X_i w(y, s) - X_i w(0, 0)| = r^{-\alpha} |X_i(u)((x, t) \circ \delta_r(y, s)) - X_i(u)(x, t)| \leq c'_2.$$

If $(\hat{x}, \hat{t}) \in C_R \setminus C_r(x, t)$ we can use the triangle inequality to obtain

$$|X_i u(x, t) - X_i u(\hat{x}, \hat{t})| \leq |X_i u(x, t) - X_i u(\tilde{x}, \tilde{t})| + |X_i u(\tilde{x}, \tilde{t}) - X_i u(\hat{x}, \hat{t})|.$$

Now, since $(x, t), (\hat{x}, \hat{t}) \in C_R \setminus \mathcal{F}$ and $(\tilde{x}, \tilde{t}) \in \mathcal{F} \cap C_R$ we can use the previous result and

$$\begin{aligned}
|X_i u(x, t) - X_i u(\hat{x}, \hat{t})| &\leq \max\{c_3, c'_3\} (d_p((x, t), (\tilde{x}, \tilde{t}))^\alpha + d_p((\tilde{x}, \tilde{t}), (\hat{x}, \hat{t}))^\alpha) \\
&\leq c''_3 d_p((x, t), (\hat{x}, \hat{t}))^\alpha,
\end{aligned}$$

which concludes the proof of (5.3).

Step 4. It remains to prove that (5.4) holds for $(x, t), (\hat{x}, \hat{t}) \in C_R \setminus \mathcal{F}$. Again, we let r and $(\tilde{x}, \tilde{t}) \in \mathcal{F} \cap C_R$ be such that $r = d_p((x, t), \mathcal{F}) = d_p((x, t), (\tilde{x}, \tilde{t}))$ and we divide the proof in two cases. Firstly we assume that $(\hat{x}, \hat{t}) \in C_R \setminus C_{r/2}(x, t)$. Then we apply the triangle inequality and

$$\begin{aligned}
&\left| u(x, t) - u(\hat{x}, \hat{t}) - \sum_{i=1}^q X_i u(\hat{x}, \hat{t})(x_i - \hat{x}_i) \right| \\
&\leq \left| u(x, t) - u(\tilde{x}, \tilde{t}) - \sum_{i=1}^q X_i u(\tilde{x}, \tilde{t})(x_i - \tilde{x}_i) \right| + \left| u(\tilde{x}, \tilde{t}) - u(\hat{x}, \hat{t}) - \sum_{i=1}^q X_i u(\hat{x}, \hat{t})(\tilde{x}_i - \hat{x}_i) \right| \\
&+ \sum_{i=1}^q |X_i(\tilde{x}, \tilde{t}) - X_i(\hat{x}, \hat{t})| \cdot |\tilde{x}_i - \hat{x}_i| \\
&\leq c'_4 (d_p((x, t), (\hat{x}, \hat{t}))^{1+\alpha} + d_p((\tilde{x}, \tilde{t}), (\hat{x}, \hat{t}))^{1+\alpha} + d_p((\tilde{x}, \tilde{t}), (\hat{x}, \hat{t}))^\alpha d_p((\tilde{x}, \tilde{t}), (x, t))) \\
&\leq c''_4 d_p((x, t), (\hat{x}, \hat{t}))^{1+\alpha}.
\end{aligned}$$

Now, assume that $(\hat{x}, \hat{t}) \in C_{r/2}(x, t) \cap C_R$. We recall that $u(x, t) - u(\hat{x}, \hat{t}) - \sum_{i=1}^q X_i u(\hat{x}, \hat{t})(x_i - \hat{x}_i)$ is in fact $u(x, t)$ minus its Taylor polynomial of δ_λ -degree 1 at (\hat{x}, \hat{t}) . In this case we use that u is a solution to (2.15) and hence $u \in \mathcal{C}_X^{2, \alpha}$, see Subsection 3. We can thus use Lemma 3.3 to find $\mathcal{C}_X^{2, \alpha}$ -bounds on u . Then we use these bounds in Theorem 2.2 to obtain the desired result. Note that we also used that we are working on a stratified group when using Theorem 2.2. This completes the proof in the case $m = 1$, $\gamma = 1 + \alpha$.

5.3 Proof of Theorem 1.1 when $m = 2$, $\gamma = 2$

To ease notation we assume, as in Lemma 3.10, that $\varphi \equiv 0$, note however that this is not restrictive. That is we assume that

$$(u, f, g, \varphi) \in \mathcal{P}_m(\mathcal{H}, \alpha, C, M_1, M_2, M_3),$$

where $\varphi \equiv 0$ and $M_3 = 0$. As before we take $R \in (0, 1)$ and $C_1 > 1$ such that $C_{2R} \subseteq C_{2C_1 R}(\hat{x}, \hat{t}) \subseteq C$ for all $(\hat{x}, \hat{t}) \in C_{2R}$ and we define $\mathcal{F} = \overline{C_{2R}} \cap \{(x, t) : u(x, t) = \varphi(x, t)\}$. We intend to prove that there exists a constant $\hat{c} = \hat{c}(\mathcal{H}, \alpha, M_1, M_2) < \infty$ such that

$$\|u\|_{S_X^\infty(C_R)} \leq \hat{c}. \quad (5.9)$$

For $(\hat{x}, \hat{t}) \in C_R \cap \{(x, t) : u(x, t) > 0\}$ we define $\hat{r} = \hat{r}(\hat{x}, \hat{t}) = \sup\{r : C_r(\hat{x}, \hat{t}) \subset C \cap \{(x, t) : u(x, t) > 0\}\}$. By the maximum principle $\mathcal{F} \cap \partial_p C_{\hat{r}}(\hat{x}, \hat{t}) \neq \emptyset$ which means that there exists $(\tilde{x}, \tilde{t}) \in C_{2R} \cap \mathcal{F} \cap \partial_p C_{\hat{r}}(\hat{x}, \hat{t})$ such that $C_{\hat{r}}(\hat{x}, \hat{t}) \subset C_{\tilde{r}}(\tilde{x}, \tilde{t})$ for some \tilde{r} , $\hat{r} < \tilde{r} < c_1 \hat{r}$, for some universal constant c_1 . Now we use Lemma 4.3 which states that for $r \in (0, \tilde{r})$ and $(x, t) \in C_r(\tilde{x}, \tilde{t}) \cap C$,

$$|u(x, t)| \leq c_2 r^2. \quad (5.10)$$

Thereafter we define, for $(x, t) \in C$,

$$v(x, t) := \frac{u^{\hat{r}, (\hat{x}, \hat{t})}(x, t)}{\hat{r}^2},$$

and in particular there holds

$$\mathcal{H}^{\hat{r}, (\hat{x}, \hat{t})} v = f^{\hat{r}, (\hat{x}, \hat{t})} \quad \text{in } C.$$

For notation, see Subsection 2.3. Furthermore, by (5.10) and (2.19),

$$\|v\|_{L^\infty(C)} \leq c_2 \quad \text{and} \quad \|f^{\hat{r}, (\hat{x}, \hat{t})}\|_{C_X^{0, \alpha}(C)} \leq M_2.$$

In particular, we can use Lemma 3.3 to conclude that

$$\|v\|_{S_X^\infty(C_{1/2}(\hat{x}, \hat{t}))} \leq \|v\|_{C_X^{2, \alpha}(C_{1/2}(\hat{x}, \hat{t}))} \leq c,$$

and (5.9) follows immediately. This completes the proof of Theorem 1.1. \square

6 Proof of Theorem 1.2

The proof is very similar to the proof of Theorem 1.1 but differs in that we, near the initial state, have to rely on Lemma 4.2 instead of Lemma 4.3. We will give a detailed proof of part *i*) of Theorem 1.2, leaving part *ii*) and *iii*). For this reason, let $R \in (0, 1)$ and $C_1 > 1$ be such that $C_{2R} \subseteq C_{2C_1 R}(\hat{x}, \hat{t}) \subseteq C$ for all $(\hat{x}, \hat{t}) \in C_{2R}$. We will now study the regularity of u near the initial state and we note that, by Subsection 2.3, it is enough to consider $\Omega = C^+$ and $\Omega' = C_R^+$. To prove part *i*) we need to show that there exists a constant $c = c\left(\mathcal{H}, \alpha, \|f\|_{C_X^{0, \alpha}(C^+)}, \|g\|_{L^\infty(C^+)}, \|\varphi\|_{C_X^{0, \alpha}(C^+)}\right)$ such that

$$\sup_{\substack{(x, t), (\hat{x}, \hat{t}) \in C_R^+ \\ (x, t) \neq (\hat{x}, \hat{t})}} \frac{|u(x, t) - u(\hat{x}, \hat{t})|}{d_p((x, t), (\hat{x}, \hat{t}))^\alpha} \leq c. \quad (6.1)$$

If $\hat{t} = 0$ in (6.1) we use Lemma 4.2 on $C_{d_p((x, t), (\hat{x}, 0))}^+(\hat{x}, 0)$ with $m = 0$ and obtain

$$\begin{aligned} |u(x, t) - u(\hat{x}, 0)| &\leq |u(x, t) - g(x, t)| + |g(x, t) - g(\hat{x}, 0)| \\ &\leq c_1 d_p((x, t), (\hat{x}, 0))^\alpha, \end{aligned} \quad (6.2)$$

since $g \in C(\overline{C^+})$. Hence (6.1) is valid if either one of t, \hat{t} vanishes and hereafter we assume that both t and \hat{t} are strictly positive. In that case we note that $d_p((x, t), (x, 0)) = \sqrt{t}$ and we will divide the rest of the proof in two cases.

Case 1: We assume that $(\hat{x}, \hat{t}) \in C_R^+ \setminus C_{\sqrt{\hat{t}/2}}(x, t)$ which implies that

$$\begin{aligned} d_p((\hat{x}, \hat{t}), (x, 0)) &\leq d_p((\hat{x}, \hat{t}), (x, t)) + d_p((x, t), (x, 0)) \\ &\leq 3d_p((\hat{x}, \hat{t}), (x, t)) \end{aligned}$$

and

$$d_p((x, t), (x, 0)) \leq 2d_p((\hat{x}, \hat{t}), (x, t)).$$

Therefore, by (6.2),

$$\begin{aligned} |u(x, t) - u(\hat{x}, \hat{t})| &\leq |u(x, t) - u(x, 0)| + |u(x, 0) - u(\hat{x}, \hat{t})| \\ &\leq d_p((x, t), (x, 0))^\alpha + d_p((\hat{x}, \hat{t}), (x, 0))^\alpha \\ &\leq 5d_p((\hat{x}, \hat{t}), (x, t))^\alpha, \end{aligned}$$

which closes case 1.

Case 2: We assume that $(\hat{x}, \hat{t}) \in C_R^+ \cap C_{\sqrt{t}/2}(x, t)$ and note that $C_{\sqrt{t}}(x, t) \subset C_{2R}^+$. By (6.2)

$$\begin{aligned} \|u - u(x, t)\|_{L^\infty(C_{\sqrt{t}}(x, t))} &\leq \|u - u(x, 0)\|_{L^\infty(C_{\sqrt{t}}(x, t))} + |u(x, t) - u(x, 0)| \\ &\leq c_2 \left(\sqrt{t}\right)^\alpha \end{aligned} \quad (6.3)$$

and we define

$$v(y, s) := \frac{u^{\sqrt{t}, (x, t)}(y, s) - u(x, t)}{(\sqrt{t})^\alpha} \quad \text{for } (y, s) \in C.$$

Using (6.3) we see that $\|v\|_{L^\infty(C)} \leq c_2$ and by using properties of v (see Subsection 2.3) and Theorem 1.1 we see that

$$|v(y, s)| \leq c_3 d_p((0, 0), (y, s))^\alpha \quad \text{for } (y, s) \in C_{1/2}.$$

By definition of v this is equivalent to

$$|u(x, t) - u(\hat{x}, \hat{t})| \leq c d_p((x, t), (\hat{x}, \hat{t}))^\alpha,$$

and this completes the proof of part *i*) of Theorem 1.2. As previously mentioned the rest of the proof, i.e., the proof of *i*) and *ii*), is much in line with the corresponding statements in Theorem 1.1 and we omit the details. This completes the proof. \square

7 Generalizations, further developments and open problems

When we proved the existence of strong solutions to the obstacle problem (1.2) in [FGN12] we were able to carry out the proofs using less restrictive assumptions than in the present paper. The main difference is that, unlike in [FGN12], we here assume that the vector fields $\{X_1, \dots, X_q\}$ are generators of the first layer of a stratified, homogeneous group. In addition we assume that $\{X_1, \dots, X_q\}$ are left invariant and homogeneous of degree one. In [FGN12] we did not have to restrict ourselves to this case and the main tool for carrying out the proofs was the lifting-approximation technique of Rotschild and Stein [RS76]. The main difficulties to overcome in the present paper, if one should try to relax these assumptions, are that of polynomial approximations and scaling, see Section 2.2 and Section 2.3 respectively. It may be possible to use the lifting-approximation technique here as well, however, it is not clear if $\mathcal{C}_X^{m, \alpha}$ estimates carry through the lifting-approximation machinery. A partial (affirmative) answer is given in [BB07] which states that if a function u is lifted to \tilde{u} and if $\tilde{u} \in \mathcal{C}_{\tilde{X}}^{m, \alpha}$, where \tilde{X} is the lifted vector field, then $u \in \mathcal{C}_X^{m, \alpha}$ as well. Whether or not a similar estimate holds for the approximation part is unclear.

Another generalization would be to consider the operator

$$\mathcal{H} = \sum_{i,j=1}^q a_{ij}(x, t) X_i X_j + \sum_{i=1}^q b_i(x, t) X_i + X_0, \quad (x, t) \in \mathbf{R}^{n+1}. \quad (7.1)$$

When we began our study in [FGN12] certain estimates for the operator in (7.1) were missing, in particular Schauder type estimates, interior \mathcal{S}^p -estimates and embedding theorems corresponding to Theorems 1.2-1.4 in [FGN12]. However, for operators (1.1) at least the necessary Schauder type estimates were established and so we chose to work with (1.1) instead of (7.1). A recent preprint by Bramanti and Zhu [BZ] has made two of the crucial estimates available also for operators (7.1). To prove the embedding theorem we need estimates on the fundamental solution. If the vector fields X_0, X_1, \dots, X_q are left invariant on a homogeneous group and if X_1, \dots, X_q are homogeneous of degree 1 while X_0 is homogeneous of degree 2 then we may use results of Folland [Fri75], in the more general case this is an open question. Provided with this estimate, and after proving the existence of strong solutions to the obstacle problem, the same method we have presented here will most likely be usable for the operator (7.1).

Finally, it would be interesting to study the regularity of the free boundary in the setting studied in the present paper.

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